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Università di Napoli Parthenope, Short Course on: "*Mathematical tools for Economic Dynamics: Dynamic Optimization*"

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Abstract. We summarize some basic result in dynamic optimization and optimal control theory, focusing on some economic applications.

Key words: Dynamic Optimization Optimal control, Dynamic Programming, Optimality conditions.

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1 Summary of the lectures

- Lecture 1. Wednesday, April 11, 2012. 15.15 - 16.15 and 16.30 - 17.30.

  Introduction to Dynamic Optimization in discrete and continuous time: example of utility maximization from static case to dynamic case.

  Basic setting of Dynamic Optimization problems as Optimal Control (OC) problems and as Calculus of Variation (CV) problems.

  Definition of Admissible and Optimal Control Strategies, Admissible and Optimal State Trajectories, Value Function.

  Open Loop and Closed Loop Control Strategies: main ideas. Definition of Admissible and Optimal Feedback Strategies. How to recover state trajectories and control strategies from a given feedback map: Closed Loop Equation.

  Examples of Dynamic Optimization in discrete and continuous time: optimal consumption and optimal investment.

- Lecture 2. Thursday, April 12, 2012. 9.30 - 11 and 11.30 - 13.

  The Dynamic Programming Method in the discrete time case: the Bellman equation and its use to find the value function and the optimal strategies in feedback form.

  The discounted infinite horizon case: new form of the Bellman equation Examples. Some ideas in the continuous time case and in the stochastic case.

  Exercises on Dynamic Programming Method in discrete time in finite and infinite horizon.

- Lecture 3. Friday, April 13, 2012. 9.30 - 11 and 11.30 - 13.

  The Dynamic Programming Method in the continuous time case: the Bellman equation and the Hamilton-Jacobi-Bellman equation and its use to find the value function and the optimal strategies in feedback form.

  Dynamic Optimization in discrete and continuous time: the Maximum Principle and application to the main examples.

2 Readings

More economically oriented books are the following (in alphabetical order).

- Daron Acemoglu, (2009),

  Introduction to Modern Economic Growth, Princeton University Press. Ch. 6, 7, 8 and 16.

- Alpha Chiang, (1992),

  Elements of Dynamic Optimization, Waveland Press.

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3 Introduction to dynamic optimization

Dynamic optimization problems are substantially optimization problems where the decisions variables and other parameters of the problem *possibly* vary with time. In fact it is not easy to give a formal definition of what dynamic optimization problems are: we will not attempt to do it. In this note we will deal with a certain class of dynamic optimization problems that arise in economic (and also in engineering, biology, finance, etc.) applications.

3.1 An example: utility maximization

The problem of utility maximization is a well known problem in economic theory. It can be formulated in a static context or in a dynamic context. We will consider a very simple case from the economic point of view to illustrate the passage from the static to the dynamic problem.

3.1.1 Static case

Let us start from the static problem in a simple case. Consider a consumer with an initial amount of money x_0 that can consume k differents good and want to maximize its satisfaction from consumption without taking any debt. If $\mathbf{c} = (c_1, ..., c_k)$ is the vector of (clearly nonnegative) consumed quantities, $\mathbf{p} = (p_1, ..., p_k)$ and $U(\mathbf{c})$ is the satisfaction from consumption ($U_0 : \mathbb{R}^k_+ \to \mathbb{R}_+$, is usually concave and increasing in each component) then the problem is the following

$$Maximize \qquad U_0(\mathbf{c}) \tag{1}$$

$$with\, the\ \ k+1\ \ constraints: \quad \mathbf{c} \geq \mathbf{0}, \quad \langle \mathbf{p}, \mathbf{c} \rangle \leq x_0.$$

Its main ingredients are the objective function to maximize U_0 and the constraints to be respected.

3.1.2 Dynamic case

A dynamic formulation of the above problem can be the following. The consumer may spread the consumption over a certain period of time: this means that the consumption vector $\mathbf{c}$ will become a time dependent vector, and so may do the price vector $\mathbf{p}$. The set of times where the decision of consumption may be taken is usually a subset $\mathcal{T}$ of the positive half line $\mathbb{R}_+ = [0, +\infty)$. How to choose the subset $\mathcal{T}$? Usually one chooses an *initial time* t_0 and a *final time* T satisfying the obvious inequalities $0 \leq t_0 \leq T \leq +\infty$. The final time T will be usually called the *horizon* of the problem: if $T < +\infty$ we speak of *finite horizon* problem, if $T = +\infty$ we speak of *infinite horizon* problem. Then there are two main possibilities:

- $\mathcal{T} = [t_0, T] \cap \mathbb{N}$, i.e. the *discrete time* case. The name "discrete" may come from the fact that, as a topological subspace of $\mathbb{R}$, $\mathcal{T}$ is endowed with the discrete topology. Usually one chooses the initial time in $\mathbb{N}$ and the final time in $\mathbb{N} \cup \{+\infty\}$.
- $\mathcal{T} = [t_0, T] \cap \mathbb{R}$, i.e. the *continuous time* case. The name "continuous" may come from the fact that $\mathcal{T}$ is a set with cardinality $\aleph_1$ (i.e. the cardinality of the continuum).

The consumption vector $\mathbf{c}$ is now a function $\mathbf{c}(\cdot)$

$$\begin{aligned} \mathbf{c}(\cdot) &: \mathcal{T} \to \mathbb{R}^k \\ &: t \to \mathbf{c}(t) \end{aligned}$$

the consumer will maximize an intertemporal utility from consumption. Under the usual hypothesis of separability, a typical form of the intertemporal utility is

- in the case of discrete time (finite or infinite horizon T)

$$U_1(\mathbf{c}(\cdot)) := \sum_{t=t_0}^{T} \beta^t U_0(\mathbf{c}(t))$$

  where $\beta \in (0, 1)$ is a (discrete) *discount factor* (meaning that the consumer take less satisfaction from a delayed consumption);

- in the case of continuous time (finite or infinite horizon T)

$$U_1(\mathbf{c}(\cdot)) := \int_{t_0}^{T} e^{-\rho t} U_0(\mathbf{c}(t))\, dt$$

  where $\rho \in (0, +\infty)$ is a (continuous) *discount factor* (meaning again that the consumer take less satisfaction from a delayed consumption: it is set differently from the discrete time case for it is simpler to treat in this form when time is continuous).

Remark 3.1 *Both the discount factors may be chosen differently: $\beta \geq 1$ or $\rho \leq 0$, arise in some economic models (see e.g. [21]) meaning that future consumption is evaluated more than (or equally to) the present one.*

Remark 3.2 *Note that in the discrete time case the vector* $\mathbf{c}(t)$ *has the dimension of a quantity and represents the amount of consumption over the period of time* $[t, t+1)$*. In the continuous time case, differently,* $\mathbf{c}(t)$ *has the dimension of a quantity over time and represents the intensity of consumption at the time* t*: we may say that in the infinitesimal amount of time* $[t, t+dt)$ *the consumed quantity is* $\mathbf{c}(t)\,dt$*.*

Consider now the constraints on the decision variable in the static case and try to see how they transfer to the dynamic case. We consider for simplicity only the discrete time case. The positivity constraints $\mathbf{c} \geq \mathbf{0}$ immediately transfer to

$$\mathbf{c}(t) \geq 0 \qquad \forall t \in \mathcal{T},$$

with obvious meaning. The budget constraint $\langle \mathbf{p}, \mathbf{c} \rangle \leq x_0$ is more difficult. In fact it needs to be satisfied at any time $t \in \mathcal{T}$ but now the amount of money in the pocket of the consumer vary with time. In the dicrete time case we have, assuming that the price vector $\mathbf{p}$ is time independent

$$\begin{aligned} \langle \mathbf{p}, \mathbf{c}(t_0) \rangle &\leq x_0, \qquad \langle \mathbf{p}, \mathbf{c}(t_0+1) \rangle \leq x_0 - \langle \mathbf{p}, \mathbf{c}(t_0) \rangle, \\ \langle \mathbf{p}, \mathbf{c}(t_0+2) \rangle &\leq x_0 - \langle \mathbf{p}, \mathbf{c}(t_0) \rangle - \langle \mathbf{p}, \mathbf{c}(t_0+1) \rangle \end{aligned}$$

and so on. In compact form we can write

$$\langle \mathbf{p}, \mathbf{c}(t) \rangle \leq x_0 - \sum_{s=t_0}^{t-1} \langle \mathbf{p}, \mathbf{c}(s) \rangle, \qquad \forall t \in \mathcal{T}.$$

Our dynamic optimization problem becomes then

$$Maximize \qquad U_1(\mathbf{c}(\cdot)) = \sum_{t=t_0}^{T} \beta^t U_0(\mathbf{c}(t)) \tag{2}$$

with the constraints :

$$\mathbf{c}(t) \geq 0 \qquad \forall t \in \mathcal{T}, \tag{3}$$

$$\langle \mathbf{p}, \mathbf{c}(t) \rangle \leq x_0 - \sum_{s=t_0}^{t-1} \langle \mathbf{p}, \mathbf{c}(s) \rangle, \qquad \forall t \in \mathcal{T}. \tag{4}$$

The constraints above are very different. The first is *static* (or, say, "Markovian") in the sense that it does not involve neither relationship between the decision variables at different periods, nor derivatives of the variables itself). The second one is *dynamic* (or, say, "NonMarkovian") in the sense that it does involve the values of the decision variables at different times. This second on is more difficult to deal with. To get rid of such difficult constraint we introduce the state variables in next section.

3.1.3 The state variables

The dynamic optimization problem above is clearly more difficult than the static problem (1). In particular the intertemporal budget constraint (4) is quite nasty since it involves the values of $\mathbf{c}(t)$ at all times between the initial time t_0 and the "current" time t. The reason for this is the fact that there is a time evolution now and, in particular, the amount of money available for the consumer vary with the time depending on the whole history of the decisions variables.

We would like to transform such constraint in a more treatable one involving only the "current" time t or, at most, t and its successive $t+1$.

To do this we introduce a new variable in the problem: the so called *state variable* $x(t)$ representing the amount of money available at the time t or, more precisely, at the beginning of the t-th period, before the consumption $\mathbf{c}(t)$ takes place.

Remark 3.3 *The name* state variable *comes from the following idea. We are dealing with an economic* system *(the pocket of the consumer) evolving with time and we want to describe it with a dynamic variable $x(\cdot)$ such that $x(t)$ well explains the* state[1] *of such system at time t. The system can be modified by an agent using the* decision variable *(called often the* control variable*) $\mathbf{c}(\cdot)$ that also influences the state $x(\cdot)$ of the system.*

Since $x(t)$ is the amount of money in the pocket of the consumer at the beginning of the t-th period it must be

$$x(t) = x_0 - \sum_{s=t_0}^{t-1} \langle \mathbf{p}, \mathbf{c}(s) \rangle, \qquad \forall t \in \mathcal{T}. \tag{5}$$

Note that, if T is finite it make sense to define also

$$x(T+1) = \sum_{s=t_0}^{T} \langle \mathbf{p}, \mathbf{c}(s) \rangle,$$

which is the amount of money left in the pocket of the consumer at the end of the last period. With this definition the constraint (4) becomes

$$\langle \mathbf{p}, \mathbf{c}(t) \rangle \leq x(t), \qquad \forall t \in \mathcal{T}. \tag{6}$$

However such constraint can be rewritten in a more convenient way as follows. It can be easily seen from (5) that

$$x(t+1) = x(t) - \langle \mathbf{p}, \mathbf{c}(t) \rangle, \qquad \forall t \in \mathcal{T}, \tag{7}$$

with the agreement that, in the finite horizon case, for $t = T$, $x(T+1)$ is defined as above. So, requiring (6) is equivalent to require[2]

$$x(t+1) \geq 0, \qquad \forall t \in \mathcal{T}. \tag{8}$$

[1] The terminology used here: system, state, etc. is probably borrowed from physics and engineering.

[2] Note that, taking account that $x(t_0) = x_0 \geq 0$ this last constraint can be rewritten as

$$x(t) \geq 0, \qquad \forall t \in \mathcal{T} \cup \{T+1\}$$

The last two constraints are enough to characterize the state variable $x(\cdot)$ and to include the constraint (4). The end of the story is then the following. We introduce the state variable

$$x(\cdot) : \mathcal{T} \to \mathbb{R}$$

as the unique solution of the difference equation (often called the *state equation*)

$$\begin{cases} x(t+1) = x(t) - \langle \mathbf{p}, \mathbf{c}(t) \rangle & \forall t \in \mathcal{T}, \\ x(t_0) = x_0, \end{cases}$$

and we require it to satisfy the constraints

$$x(t+1) \geq 0, \qquad \forall t \in \mathcal{T}.$$

The problem (2) may now be rewritten as follows

$$Maximize \qquad U_1(\mathbf{c}(\cdot)) = \sum_{t=t_0}^{T} \beta^t U_0(\mathbf{c}(t)) \tag{9}$$

with the constraints :

$$\begin{array}{rcll} \mathbf{c}(t) & \geq & 0 \quad \forall t \in \mathcal{T}, & \text{(PC1)}, \\ x(t) & \geq & 0, \quad \forall t \in \mathcal{T}, & \text{(PC2)}, \end{array}$$

$$\begin{cases} x(t+1) = x(t) - \langle \mathbf{p}, \mathbf{c}(t) \rangle & \forall t \in \mathcal{T}, \\ x(0) = x_0, \end{cases} \qquad \text{(SE)}.$$

Note that here we still have a dynamic constraint (the state equation). However such constraint involve only the values of $x(\cdot)$ at the current time t and its successive $t+1$ (not involving the whole history as in (4)) and it is used to define the evolution of the state variable $x(\cdot)$. This makes it more treatable.

The above is a classical discrete time Optimal Control Problem (OCP) whose main ingredients are:

- the functional U_1 to maximize;
- the dynamic constraint (SE) involving both the state and control variable which is formulated as a difference equation (which will be a differential equation in continuous time) where the state variable is the unknown and the control variable is a parameter to be chosen. This yields the name of *state equation*. and the acronym (SE) used to denote it.

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in the finite horizon case and

$$x(t) \geq 0, \qquad \forall t \in \mathcal{T}$$

in the infinite horizon case.

- The static constraints (PC1) and (PC2) which involve the state and/or control variable only at the current time t. Such constraints may involve only the control variable (*control constraints*) as (PC1) or only the state variable (*state constraints*) as (PC2) or both (*state-control constraints*). We will use the symbol (PC) (pointwise constraints) to denote them.

3.1.4 The continuous time formulation

We can now provide the analogous continuous time formulation of the above problem. The state equation describing the dynamic evolution of the state variable is now given by an ordinary differential equation (ODE):

$$\begin{cases} x'(t) = -\langle \mathbf{p}, \mathbf{c}(t) \rangle & \forall t \in \mathcal{T}, \\ x(t_0) = x_0, \end{cases}$$

while the pointwise constraints are unchanged. The problem then becomes

$$Maximize \qquad U_1(\mathbf{c}(\cdot)) = \int_{t_0}^{T} e^{-\rho t} U_0(\mathbf{c}(t))\, dt$$

with the constraints :

$$\begin{array}{rcll} \mathbf{c}(t) & \geq & 0 \quad \forall t \in \mathcal{T}, & \text{(control constraints)}, \\ x(t) & \geq & 0, \quad \forall t \in \mathcal{T}, & \text{(state constraint)}, \end{array}$$

$$\begin{cases} x'(t) = -\langle \mathbf{p}, \mathbf{c}(t) \rangle & \forall t \in \mathcal{T}, \\ x(t_0) = x_0, \end{cases} \qquad \text{(state equation)}.$$

3.1.5 A special case: the Gale's cake (cake eating)

The problem of the Gale's cake (cake eating) is a special case of the above one. We consider a cake to be consumed in the period $[t_0, T]$. The state variable $x(t)$ is the amount of cake remained. The control variable c (only one: here $k = 1$) is the amount of cake consumed in the period $[t, t+1)$. The precise formulation is then the same as above (see (9)) when $k = 1$ and $\mathbf{p} = p = 1$. Then

$$Maximize \qquad U_1(c(\cdot)) = \sum_{t=t_0}^{T} \beta^t U_0(c(t))$$

with the constraints :

$$\begin{array}{rcll} c(t) & \geq & 0 \quad \forall t \in \mathcal{T}, & \text{(control constraint)}, \\ x(t) & \geq & 0, \quad \forall t \in \mathcal{T}, & \text{(state constraint)}, \end{array}$$

$$\begin{cases} x(t+1) = x(t) - c(t) & \forall t \in \mathcal{T}, \\ x(t_0) = x_0, \end{cases} \qquad \text{(state equation)}.$$

Remark 3.4 *Observe that in this case, differently from the one above, we may eliminate the variable c in the optimization problem and simply maximize with respect to the state variable x. In fact*

$$c(t) = x(t) - x(t+1) \qquad \forall t \in \mathcal{T}$$

and the control constraint $c(t) \geq 0$ may be written

$$x(t) - x(t+1) \geq 0, \qquad \forall t \in \mathcal{T},$$

i.e. the sequence x is decreasing. Then the problem becomes

$$Maximize \qquad U_2(x(\cdot)) := \sum_{t=t_0}^{T} \beta^t U_0(x(t) - x(t+1)) \tag{10}$$

with the constraints :

$$\begin{array}{rcll} x(t) - x(t+1) & \geq & 0 & \forall t \in \mathcal{T}, \\ x(t) & \geq & 0, & \forall t \in \mathcal{T}, \\ x(0) & = & x_0. & \end{array} \tag{11}$$

This formulation is the classical one for the problems of Calculus of Variation (CV). By setting $X = [0, +\infty)$ and $D(x) = [0, x]$ for every $x \in X$. we can rewrite the constraints (11) in the form

$$\begin{array}{rcll} x(t) & \in & X & \forall t \in \mathcal{T}, \\ x(t+1) & \in & D(x(t)) & \forall t \in \mathcal{T}. \end{array}$$

The optimization problem in the form (10) - (11) it is the standard problem studied in the books of Montrucchio [29] and Stokey - Lucas [38].

Similarly we can provide the continuous time formulation of the Gale's cake problem

$$Maximize \qquad U_1(c(\cdot)) = \int_{t_0}^{T} e^{-\rho t} U_0(c(t))\, dt$$

with the constraints :

$$\begin{array}{rcll} c(t) & \geq & 0 & \forall t \in \mathcal{T}, \qquad \text{(control constraint)}, \\ x(t) & \geq & 0, & \forall t \in \mathcal{T}, \qquad \text{(state constraint)}, \end{array}$$

$$\begin{cases} x'(t) = -c(t) & \forall t \in \mathcal{T}, \\ x(t_0) = x_0, & \end{cases} \qquad \text{(state equation)}.$$

Remark 3.5 *Also in the continuous time Gale's cake problem we may eliminate the variable c in the optimization problem and simply maximize with respect to the state variable x. In fact*

$$c(t) = -x'(t), \qquad \forall t \in \mathcal{T}$$

and the control constraint $c(t) \geq 0$ may be written

$$x'(t) \leq 0, \qquad \forall t \in \mathcal{T},$$

i.e. the function x is decreasing. Then the problem becomes

$$Maximize \qquad U_2(x(\cdot)) := \int_{t_0}^{T} e^{-\rho t} U_0(-x'(t))\, dt$$

with the constraints :

$$\begin{aligned} x'(t) &\leq 0 \quad \forall t \in \mathcal{T}, \\ x(t) &\geq 0, \quad \forall t \in \mathcal{T}, \\ x(0) &= x_0. \end{aligned}$$

This formulation is the classical one for the problems of Calculus of Variation (CV) (see e.g. [24]).

3.2 Optimal control problems

Optimal control problems (OCP) are more or less optimization problems, i.e. looking for the maximum or minimum of certain functions, "coupled" with a control system. By *control system* we mean a physical (or economic, or else) system whose behavior is described by a state variable and that can be controlled from outside by an input (or control) variable. Generally the state and control variable are required to satisfy an ordinary differential equation (ODE) in the continuous time case or a difference equation (DE) in the discrete time case. In optimal control one wants to optimize the behavior of the system by maximizing or minimizing a given function of the state and control variables. In most cases (but not all: an example are the problems with incentive, see e.g. [13]) the controlled system evolves with time, so its behavior is described by an evolution equation and we will call it a *controlled dynamical system*. The optimal control problem in such cases is often called a *dynamic optimization problem*.

Here we want to deal with a class of optimal control problems that includes some important economic examples. Main goal are:

- *give an abstract formulation of a wide class of optimal control problems;*
- *give a brief outline of the main methods used to treat such problems.*
- *show how to apply such methods to some economic examples like the ones described above.*

3.2.1 A general formulation

The state equation, the pointwise constraints and the set of admissible strategies We fix $T \in [0, +\infty]$ (final time or horizon) and $t_0 \in [0, T]$ (initial time). Then we take two sets: $C \subseteq \mathbb{R}^k$ (the set of control variables) and $X \subseteq \mathbb{R}^n$ (the set of state variables) and fix a point $x_0 \in X$ (the initial state). A *control strategy* will be a function $c(\cdot) : \mathcal{T} \to C$, (usually in the continuous case one requires at least local integrability of $c(\cdot)$) while a *state trajectory* will be a function $x(\cdot) : \mathcal{T} \to X$ (usually in the continuous case x has to be continuous). Given the control strategy $c(\cdot)$ the state trajectory is determined by a *state equation (SE)* which describes the time evolution of the system. We consider state equation of the following kind: given a map $f_D : \mathcal{T} \times X \times C \mapsto X$ (discrete time case) or $f_C : \mathcal{T} \times X \times C \mapsto \mathbb{R}^n$ (continuous time case), and a control strategy $c(\cdot)$ the state trajectory is the unique solution[3] of the difference equation (in the discrete time case)

$$\begin{cases} x(t+1) = f_D(t, x(t), c(t)); & t \in \mathcal{T} = [t_0, T] \cap \mathbb{N} \\ x(t_0) = x_0 & x_0 \in X \subseteq \mathbb{R}^n, \end{cases}$$

or of the ODE (in the continuous time case)

$$\begin{cases} x'(t) = f_C(t, x(t), c(t)); & t \in \mathcal{T} = [t_0, T] \cap \mathbb{R} \\ x(t_0) = x_0 & x_0 \in X \subseteq \mathbb{R}^n. \end{cases}$$

This is what is called a *controlled dynamical system*. The unique solution of the SE, given the initial data (t_0, x_0) and the control strategy $c(\cdot)$ will be denoted by $x(\cdot; t_0, x_0, c(\cdot))$ or simply by $x(\cdot)$ when no confusion may arise.

Moreover we consider some additional pointwise (or static) constraints (PC) on the state-control variables. Given two functions $g : \mathcal{T} \times X \times C \to \mathbb{R}^p$ and $h : \mathcal{T} \times X \times C \to \mathbb{R}^q$ we ask that

$$g(t, x(t), c(t)) \leq 0, \quad \forall t \in \mathcal{T}, \tag{12}$$

and

$$h(t, x(t), c(t)) \leq 0, \quad \forall t \in \mathcal{T}_1 \subset \mathcal{T}.$$

The difference is that the first constraints hold for every time, while the second hold only for certain times (tipically the last one).

[3]Of course this means that suitable assumptions on the dynamics f_D or f_C are needed to ensure existence and uniqueness of solutions. For the discrete time case it is enough to ask that f_D take values in X, as written above. A standard assumptions for the continuous case is that f_C is continuous and there exists a constant $M > 0$ such that

$$||f_C(t, x, u)||_{\mathbb{R}^n} \leq M(1 + ||x||_{\mathbb{R}^n} + ||u||_{\mathbb{R}^k})$$

$\forall (t, x, u) \in \mathcal{T} \times X \times U$ and

$$||f_C(t, x_1, u) - f(t, x_2, u)||_{\mathbb{R}^n} \leq M ||x_1 - x_2||_{\mathbb{R}^n}$$

$\forall (t, u) \in \mathcal{T} \times U$ and $x_1, x_2 \in X$.

The set of all control strategies c such that the associated state-control couple $(c(\cdot), x(\cdot; t_0, x_0, c))$ satisfy all the above given constraints will be called the set of admissible strategies and denoted by the symbol $\mathcal{C}_{\text{ad}}(t_0, x_0)$ underlying its dependence on the initial data. In symbols, for the discrete time case:

$$\mathcal{C}_{\text{ad}}(t_0, x_0) = \left\{ \begin{array}{c} c(\cdot) : \mathcal{T} \to C \text{ such that } \quad x(t) \in X \text{ and } g(t, x(t), c(t)) \leq 0, \quad \forall t \in \mathcal{T}; \\ h(t, x(t), c(t)) \leq 0, \quad \forall t \in \mathcal{T}_1 \subset \mathcal{T} \end{array} \right\};$$

for the continuous time case the only difference is that one requires some regularity on $c(\cdot)$. A typical case is when we ask local integrability:

$$\mathcal{C}_{\text{ad}}(t_0, x_0) = \left\{ \begin{array}{c} c(\cdot) \in L^1_{loc}(\mathcal{T}; C) \text{ such that } \quad x(t) \in X \text{ and } g(t, x(t), c(t)) \leq 0, \quad \forall t \in \mathcal{T}; \\ h(t, x(t), c(t)) \leq 0, \quad \forall t \in \mathcal{T}_1 \subset \mathcal{T} \end{array} \right\}.$$

Remark 3.6 *Observe that the sets X and C may also be defined by constraints like (12). So one may always decide to keep $X = \mathbb{R}^n$ and $C = \mathbb{R}^k$. This depends on the single problem.*

The objective function The objective of the problem is to maximize a given functional $J(t_0, x_0; c(\cdot))$ [4] over the set the admissible strategies. In general J will be defined on a greater set $\mathcal{C}$ of control strategies (e.g. $\mathcal{C} = L^1_{loc}(\mathcal{T}; C)$ in a typical continuous time case or $\mathcal{C} = C^{\mathcal{T}}$ in a typical discrete time case) but the interest is limited to $\mathcal{C}_{\text{ad}}$. We now provide a class of functionals that is commonly used. Two functions

$$f_0 : \mathcal{T} \times X \times C \to \mathbb{R},$$

$$\phi_0 : X \to \mathbb{R}$$

(usually at least measurable) are given, denoting respectively the instantaneous performance index of the system and the payoff from the final state[5]. The typical functionals are given below separating the various cases.

- In the discrete time case a typical form of the functional to maximize is

$$J(t_0, x_0; c(\cdot)) = \sum_{t=t_0}^{T} f_0(t, x(t), c(t)) + \phi_0(x(T+1))$$

  for the finite horizon case[6] and

$$J(t_0, x_0; c(\cdot)) := \sum_{t=t_0}^{+\infty} f_0(t, x(t), c(t))$$

[4] We write in this way to underline the dependence on initial parameters. The control variables are separated from the parameters by a ; sign.

[5] The term ϕ_0 measuring the payoff from the final state is not usually considered in the infinite horizon case

[6] Note that, as in the example of utility maximization, $x(T+1)$ is the value of the state variable after the end of the control period and is defined by the state equation when $t = T$.

for the infinite horizon case. Tipically in the infinite horizon case the function f_0 is of the form

$$f_0(t, x(t), c(t)) = \beta^t f_1(x(t), c(t))$$

where $\beta \in (0, +\infty)$ is a (discrete) *discount factor* and $f_1 : X \times C \to \mathbb{R}$ is given.

- in the continuous time case

$$J(t_0, x_0; c(\cdot)) = \int_{t_0}^{T} f_0(t, x(t), c(t)dt + \phi_0(x(T))$$

for the finite horizon case and

$$J(t_0, x_0; c(\cdot)) := \int_{t_0}^{+\infty} f_0(t, x(t), c(t))\, dt$$

for the infinite horizon case. Tipically in the infinite horizon case the function f_0 is of the form

$$f_0(t, x(t), c(t)) = e^{-\rho t} f_1(x(t), c(t))$$

where $\rho \in \mathbb{R}$ is a (continuous) *discount factor* and $f_1 : X \times C \to \mathbb{R}$ is given.

The problem is then

$$(P) \qquad \begin{array}{c} Maximize \quad J(t_0, x_0; c) \\ over \quad c \in \mathcal{C}_{\text{ad}}(t_0, x_0) \end{array}$$

Remark 3.7 *Often, given an OCP arising in an economic model in discrete time, one is interested to find the "equivalent" continuous time problem (or viceversa). In this case:*

- *First one has to be careful about the meaning of the word "equivalent". In fact passing from discrete time to continuous time always significatively alter the results on the model (see e.g. the discussion in [14]).*
- *Second, if the SE in the discrete case is*

$$x(t+1) = f_D(t, x(t), c(t))$$

then the analogous contuinuous time SE is

$$x'(t) = f_D(t, x(t), c(t)) - x(t)$$

so $f_C(t, x, c) = f_D(t, x, c) - x$ and the control variable change its dimensionality (see Remark 3.2). In fact the discrete time analogous of the derivative $x'(t)$ is the incremental ratio $\frac{x(t+1)-x(t)}{1}$ along the "informal" steps

$$\frac{x(t+1) - x(t)}{1} \quad \longrightarrow \quad \frac{x(t+\Delta t) - x(t)}{\Delta t} \quad \longrightarrow \quad x'(t)$$

so we may write, again informally

$$x(t+1) = f_D(t,x(t),c(t)) \longrightarrow x(t+1)-x(t) = f_D(t,x(t),c(t))-x(t)$$

$$\longrightarrow \frac{x(t+1)-x(t)}{1} = f_D(t,x(t),c(t))-x(t)$$

$$\longrightarrow x'(t) = f_D(t,x(t),c(t))-x(t)$$

Remark 3.8 *We always consider maximization problem here. Recalling that, for a given function F*

$$\max F = -\min(-F)$$

we can treat with the same ideas also minimization problems.

3.2.2 Some useful definitions and remarks

Let us give two useful definitions.

Definition 3.9 *A control strategy $c^*(\cdot) \in \mathcal{C}_{ad}(t_0,x_0)$ is an* optimal control strategy *at the starting point (t_0,x_0) if $J(t_0,x_0;c^*(\cdot)) \geq J(t_0,x_0;c(\cdot))$ for every other admissible strategy $c(\cdot) \in \mathcal{C}_{ad}(t_0,x_0)$. The corresponding state trajectory $x(\cdot;t_0,x_0;c^*)$ is an* optimal state trajectory *and will be often denoted simply by $x^*(\cdot)$. The state-control couple $(x^*(\cdot),c^*(\cdot))$ will be called an* optimal couple*.*

Definition 3.10 *The* value function *of the problem (P) is defined as*[7]

$$V(t_0,x_0) \stackrel{def}{=} \sup_{c\in\mathcal{C}_{ad}(t_0,x_0)} J(t_0,x_0;c(\cdot)).$$

Remark 3.11 *We observe that the definition of optimal control strategy at (t_0,x_0) make sense if we know that the value function is finite at that point. Of course, it can happen that $v=+\infty$ or $-\infty$. This is the case for example in many problems with infinite horizon arising in economic applications, including also the example in subsection (4.1) for some values of the parameters. In these cases one has to introduce a more general concept of optimality (see e.g. [37, Section 3.7] or [27, Ch. 9]). We avoid to treat this case for simplicity.*

We will then work with the following assumption.

Hypothesis 3.12 *The value function V is always finite.*

[7]Such function depends also on the horizon T but, for our purposes we underline only its dependence on the initial data (t_0,x_0).

Remark 3.13 *This is guaranteed for example in the discrete time finite horizon case when we know that* $\mathcal{C}_{ad}(t_0, x_0) \neq \emptyset$ *for every* $(t_0, x_0) \in \mathcal{T} \times X$*. In the discrete time infinite horizon case we have also to ask that, for example:*

- *the series in* J *is always regular and* $J(t_0, x_0; c(\cdot)) < +\infty$ *for every* $c(\cdot) \in \mathcal{C}_{ad}(t_0, x_0)$ *(is never divergent to* $+\infty$*);*
- *for every* $R > 0$ *there exists a constant* $C_R > 0$ *such that*

$$J(t_0, x_0; c(\cdot)) \leq C_R \qquad \forall (t_0, x_0) \quad \text{with} \quad |x_0| \leq R, \quad \forall c(\cdot) \in \mathcal{C}_{ad}(t_0, x_0).$$

The first conditions is necessary to have the value function finite. The second is sufficient but not necessary.

Remark 3.14 *If we consider the discrete time discounted infinite horizon problem where* $f_0 = \beta^t f_1$ *and* fD, g *are autonomous, so*

$$J(t_0, x_0, c(\cdot)) = \sum_{t=0}^{+\infty} \beta^t f_1(x(t), c(t))\, dt$$

with state equation

$$\begin{cases} x(t+1) = f_D(x(t), c(t)); & t \in \mathcal{T} = [t_0, +\infty) \\ x(t_0) = x_0 & x_0 \in X \subseteq \mathbb{R}^n, \end{cases} \tag{13}$$

and constraints

$$x(t) \in X, \quad c(t) \in C, \quad \forall t \in \mathcal{T}$$
$$g(x(t), c(t)) \leq 0, \quad \forall t \in \mathcal{T},$$

then we have that, for every $(t_0, x_0) \in \mathcal{T} \times X$

$$V(t_0, x_0) = \beta^{t_0} V(0, x_0).$$

So in this case it is enough to know the function $V_0(x_0) = V(0, x_0)$ *to calculate the value function.*

Similarly, in the continuous time case with discounted infinite horizon, if where $f_0 = e^{\rho t} f_1$ *and* fC, g *are autonomous, we have*

$$V(t_0, x_0) = e^{\rho t_0} V(0, x_0).$$

So also in this case it is enough to know the function $V_0(x_0) = V(0, x_0)$ *to calculate the value function.*

Now let say two words about the main goals we have in studying a control problem. This depends of course on the nature of the problem. Usually, in problems arising in economics one is interested in calculating, or at least study the properties, of optimal state-control trajectories $(x(\cdot), c(\cdot))$ starting at a given point (t_0, x_0). In particular, in studying problems of growth theory, the time horizon T is set equal to $+\infty$ and the main interest are the asymptotic properties of optimal trajectories (plus their rate of convergence).

Remark 3.15 *We observe that in the above formulation we considered the functional J depending only on the control c and on the initial data (t_0, x_0). Of course we could also see J as a functional defined on the state control couple $(x(\cdot), c(\cdot))$ satisfying the equation (13) as a constraint. This formulation can be useful in some case, in particular in looking for optimality conditions. In fact in this way the optimal control problem (P) takes the form of a constrained optimization problem in infinite dimension and optimality conditions can be found by using a generalized multiplier rule, in analogy with the finite dimensional case. For example, if we consider the case of discrete time infinite horizon with autonomous constraints, $h = 0$, $t_0 = 0$ and discounted objective functional, then we could write the problem (P) as follows.*

Maximize the functional

$$J\left(x\left(\cdot\right), c\left(\cdot\right)\right) = \sum_{t=0}^{+\infty} \beta^t f_1\left(x\left(t\right), c\left(t\right)\right) dt$$

under the constraints

$$x\left(t\right) \in X, \quad c\left(t\right) \in C, \quad \forall t \in \mathcal{T},$$

$$\begin{cases} x\left(t+1\right) = f_D\left(x\left(t\right), c\left(t\right)\right), & \forall t \in \mathcal{T}, \\ x\left(0\right) = x_0, & \\ g\left(x\left(t\right), c\left(t\right)\right) \leq 0, & \forall t \in \mathcal{T}, \end{cases}$$

This may be seen as a standard form of the problem (P) in the calculus of variation (CV) setting. Moreover, by eliminating, if possible[8] *the control variables, the problem may be rewritten in the form (see [29], [38])*

(P_1) **Maximize the functional**

$$J\left(x_0, x\left(\cdot\right)\right) = \sum_{t=0}^{+\infty} \beta^t f_2\left(x\left(t\right), x\left(t+1\right)\right) dt$$

under the constraints

$$\begin{array}{rcll} x\left(t\right) & \in & X & \forall t \in \mathcal{T}, \\ x\left(0\right) & = & x_0 & \\ x\left(t+1\right) & \in & D\left(x\left(t\right)\right) & \forall t \in \mathcal{T}. \end{array}$$

where the multivalued application $D : X \to X$ (which can be seen as an application $D : X \to \mathcal{P}\left(X\right)$ where $\mathcal{P}\left(X\right)$ denotes the set of all subsets of X) can be defined as

$$D\left(x\right) = \left\{y \in X : \quad \exists c \in C \text{ with } y = f_D\left(x, c\right) \text{ and } g\left(x, c\right) \leq 0\right\}.$$

[8] In fact this is always possible eventually adding new state variables to the problem.

3.2.3 Feedback control strategies

The concept of feedback strategy plays a crucial role in optimal control theory. The idea of feedback is just the one of looking at the system at any time $t \in \mathcal{T}$, observe its state $x(t)$ and then choose in real time the control strategy $c(t)$ as a function of the state (and maybe of the time) at the same time, so that

$$c(t) = G(t, x(t))$$

for a suitable map $G : \mathcal{T} \times X \to C$. A key point is that the form of G *does not depend on the initial time and state*, this is more or less obvious in the philosophy of "controlling in real time". To be more precise we introduce the following concepts.

Definition 3.16 *A function $G : \mathcal{T} \times X \to C$ is called an admissible feedback map for problem (P) if, for any initial data (t_0, x_0) the* closed loop equation*, which is*

$$\begin{cases} x(t+1) = f_D(t, x(t), G(t, x(t))); & \forall t \in \mathcal{T} \\ x(t_0) = x_0 \end{cases}$$

for the discrete time case and

$$\begin{cases} x'(t) = f_C(t, x(t), G(t, x(t))); & \forall t \in \mathcal{T} \\ x(t_0) = x_0 \end{cases}$$

for the continuous time case, admits a unique solution[9] *denoted by $x(\cdot; t_0, x_0, G)$ and the corresponding control strategy*

$$c_{(t_0,x_0,G)}(t) = G(t, x(t; t_0, x_0, G)) \qquad \forall t \in \mathcal{T}$$

is admissible, i.e. it belongs to $\mathcal{C}_{ad}(t_0, x_0)$.

An admissible control strategy for problem (P) is usually called an *"open loop" control strategy.* An admissible feedback map G will be called *"closed loop" control strategy.*

Definition 3.17 *An admissible feedback map G is optimal for problem (P) if, for every initial data (t_0, x_0) the state-control couple $\left(x(\cdot; t_0, x_0, G), c_{(t_0,x_0,G)(\cdot)}\right)$ is optimal in the sense of Definition 3.9.*

Remark 3.18 *It is possible to study the control problem (P) by restricting the admissible strategy only to closed loop strategies G with a given regularity. In general this would restrict the set of admissible control strategies, but in many cases the supremum is the same, so it is equivalent to look for optimal open loop or closed loop strategies. We do not go deep into this point, observing only that, given a problem (P), if we show that there exists an optimal feedback strategy, we have in fact proven the equivalence. This will be an outcome of Dynamic Programming Method in Section 6.*

[9] This obvious in discrete time, much less obvious in continuous time.

Remark 3.19 *Observe that, if we know the optimal strategy in closed loop form we are able to control the system in real time without knowing the real input map* $c_{(t_0,x_0,G)}(\cdot)$. *In fact it is enough to know* G *and to put a "feedback device" that reads the state* $x(t)$ *and give the value* $c(t)$ *at any time* t. *This is a common technique in many real systems (especially in engineering).*

Remark 3.20 *The two philosophies of open loop and closed loop control strategies are substantially different mostly for their different use of the information. Looking for open loop strategies means that at the starting point we look at the problem, assuming to have a perfect foresight on the future and we choose once for all the optimal strategy without changing it. On the other side, looking for closed loop strategies means that we adjust at any time our policy, depending on our observation of the system. This is clearly a better policy in the sense of the use of information and we have equivalence of the two methods if we are in a deterministic world with perfect foresight. See [22] for more considerations on this.*

4 Examples: discrete time case

4.1 Example 1: utility maximization

We consider an agent (tipically called the "consumer") that have his/her money in the bank at a fixed interest rate $r > 0$ and that uses them to "consume" at every time a quantity $c(t)$ that he can choose at any time $t \in \mathcal{T} := [0,T] \cap \mathbb{N}$ where T is the time horizon of the agent (e.g. life expectance), $T \in [0,+\infty]$. This means that, denoting by $k(t)$ the amount of money at time $t \in \mathcal{T}$ and by k_0 the initial amount of money, the function $k : \mathcal{T} \to \mathbb{R}$ satisfies the following difference equation (which we call state equation considering k as the state variable of the system)

$$k(t+1) = (1+r)\,k(t) - c(t),\ t \in [0,T]\,; \qquad k(0) = k_0. \tag{14}$$

Of course for any given control strategy c there exists a unique solution of equation (14) that we denote by $k(\cdot; k_0, c)$.

Moreover we assume for simplicity that no borrowing is allowed, so also $k(t) \geq 0$ at any $t \in \mathcal{T}$. To model the behavior of this consumer we suppose that she/he wants to maximize a certain function U (which should measure the satisfaction of the agent) of the consumption path $c(\cdot)$ up to the time horizon $T \in [0,+\infty]$. Usually such function U (called the intertemporal discounted utility from consumption) is given by

$$U(k;c) = \sum_{t=0}^{T} \beta^t u(c(t)) + \beta^t \phi(k(T+1)), \qquad \text{for } T < +\infty$$

$$U(c) = \sum_{t=0}^{+\infty} \beta^t u(c(t)), \qquad \text{for } T = +\infty$$

where $\beta \in (0,1)$ is the so-called subjective discount rate of the agent, while $u : [0,+\infty) \mapsto \mathbb{R}^+$ is the instantaneous utility from consumption and $\phi : [0,+\infty) \to \mathbb{R}^+$ is the utility from the remaining capital stock. We recall that the functions u and ϕ are generally choosen strictly increasing, concave (for economic reasons) and two times differentiable (for simplicity reasons). The standard choice of u is the so-called C.E.S. (Constant Elasticity of Substitution) utility function which is given by

$$u_\sigma(c) = \frac{c^{1-\sigma}-1}{1-\sigma} \qquad for \quad \sigma > 0,\ \sigma \neq 1$$

$$u_1(c) = \ln c \qquad for \quad \sigma = 1.$$

(we observe that, in the case $\sigma > 0,\ \sigma \neq 1$ one generally drops for simplicity the constant $(1-\sigma)^{-1}$ in the function u_σ, without changing the optimal strategies of the problem).

Summarizing, for the case $T = +\infty$, we have the following maximization problem (P): fixed the initial endowment k_0, we maximize the intertemporal discounted utility

$$U_\sigma(c) = \sum_{t=0}^{+\infty} \beta^t u_\sigma(c(t))$$

over all consumption strategies $c \in \mathcal{C}(k_0)$ where

$$\mathcal{C}(k_0) = \left\{ c \in \mathbb{R}^{\mathcal{T}} :\ c(t) \geq 0, \quad k(t;k_0,c) \geq 0 \quad \forall t \in \mathcal{T} \right\}$$

(i.e. c admissible starting at k_0). The value function of the problem is:

$$V(k_0) = \sup_{c(\cdot) \in \mathcal{A}(k_0)} \sum_{t=0}^{+\infty} \beta^t u_\sigma(c(t)).$$

Remark 4.1 *This problem has also an important meaning as a macroeconomic model where the agent is a representative agent of a certain community (see [6]). In this case the equation for the capital stock (i.e. the "state" equation) is substituted by the more general one*

$$k(t+1) = F(k(t)) + (1-\delta)k(t) - c(t),\ t \in \mathcal{T}; \qquad k(0) = k_0. \tag{15}$$

where $F(k)$ stands for the production function of the model (usually increasing and concave) and δ is the capital's depreciation factor (see e.g. [29, Chapter 1]).

Exercise 4.2 *Write this last problem in the standard form, identifying the functions f_D, f_0, f_1 and g. Show that in this case the problem can be written in the form (P_1) in Remark 3.15. Write it in such form identifying the function f_2 and the set D.*

4.2 Example 2: optimal investment

Consider the following problem of optimal investment. The equation describing the capital accumulation process is

$$\begin{cases} k(t+1) = (1-\mu)\, k(t) + I(t); & t \in \mathbb{N} \\ k(0) = k_0 & k_0 \in \mathbb{R}^+ \end{cases}$$

where $k(t)$ represents the capital stock at time t, $I(t)$ the investment at time t (i.e in the period $[t, t+1)$) and μ is the depreciation factor of capital. We assume that $k\,(t) \geq 0$ for every $t \in \mathbb{N}$ while $I \in [I_1, I_2]$ may also be negative (no irreversibility of investments). The objective is to maximize the intertemporal profit

$$J(k_0; I) = \sum_{t=0}^{+\infty} \beta^t \left[F(t, k(t)) - C(t, I(t))\right]$$

over all control strategies I belonging to the admissible set

$$\mathcal{I}_{\text{ad}}\,(k_0) = \{I : \mathbb{N} \to \mathbb{R} :\ I\,(t) \in [I_1, I_2] \quad \text{and} \quad k\,(t; 0, k_0, I) \geq 0 \quad \forall t \in \mathbb{N}\}\,.$$

Here β is the intertemporal discount factor (if we are taking simply the profit then we should take $\beta = 1 + r$, where r is the interest rate). The function $F\,(t, k)$ give the profit at time t when the capital stock is k. The function $C\,(t, I)$ gives the cost at time t of the investment I. To simplify things we may assume that F is linear (constant returns to scale) and autonomous so that $F(t, k) = F\,(k) = ak$ (a will be called "coefficient of production") and that C is quadratic and autonomous so $C(t, I) = C\,(I) = bI + cI^2$ (the linear part give the unit cost while the quadratic ones give the so-called adjustment costs). The objective functional becomes then

$$J(k_0; u) = \sum_{t=0}^{+\infty} \beta^t \left[ak(t) - bI(t) - cI^2(t)\right] dt$$

and the value function is defined as

$$V(k_0) = \sup_{I \in \mathcal{U}_{\text{ad}}} J(k_0; I).$$

Exercise 4.3 *Write this last problem in the standard form, identifying the functions f_D, f_0, f_1 and g. Show that in this case the problem can be written in the form (P_1) in Remark 3.15. Write it in such form identifying the function f_2 and the set D.*

4.3 Example 3: three finite horizon problems of optimal consumption

1. State equation:

$$\begin{cases} k(t+1) = k\,(t)\,(1 - c\,(t))\,; & t \in [t_0, T] \cap \mathbb{N} \\ k(t_0) = k_0 & k_0 \geq 0 \end{cases}$$

Maximize ($\alpha \in (0,1)$, $a > 0$):

$$J(t_0, k_0; c(\cdot)) = \sum_{t=t_0}^{T-1} [c\,(t)\, k(t)]^{\alpha} + ak(T)^{\alpha}$$

2. State equation ($\alpha \in (0,1)$):

$$\begin{cases} k(t+1) = k\,(t)^{\alpha} - c\,(t)\,; & t \in [t_0, T] \cap \mathbb{N} \\ k(t_0) = k_0 & k_0 \geq 0 \end{cases}$$

Maximize ($\beta \in (0,1)$, $b > 0$)

$$J(t, k_0; c(\cdot)) = \sum_{t=t_0}^{T-1} \beta^t \ln c\,(t) + bk(T)^{\alpha}.$$

3. Gale's cake problem with utility ($\alpha \in (0,1)$)

$$U_0\,(c) = \frac{c^{\alpha}}{\alpha}$$

Exercise 4.4 *Consider these three problems with $t_0 = 0$ and $T = 1$. Then find the optimal couple and the value function using the Kuhn-Tucker method.*

Consider these three problems with $t_0 = 0$ and $T = 2$. Then find the optimal couple and the value function using the Kuhn-Tucker method.

5 Examples: continuous time case

5.1 Example 1: utility maximization

We consider an agent (tipically called the "consumer") that have his/her money in the bank at a fixed interest rate $r > 0$ and that uses them to "consume" at a certain rate $c\,(t)$ that he can choose at any time $t \in [0, T]$ where T is the time horizon of the agent (e.g. life expectance), $T \in [0, +\infty]$. This means that, denoting by $k\,(t)$ the amount of money at time $t \in [0, T]$ and by k_0 the initial amount of money, the function $k :\in [0, T] \longmapsto \mathbb{R}$ satisfies the following differential equation (which we call "state" equation considering k as the state variable of the system)

$$\dot{k}(t) = rk(t) - c(t),\ t \in [0, T]\,; \qquad k(0) = k_0. \tag{16}$$

Of course we have to set some constraints on this equation: the consumption rate $c(t)$ needs to be nonnegative and such that the above equation (16) is solvable: so we assume that $c \in L^1_{loc}(0, +\infty; \mathbb{R}) :\ c \geq 0\ a.e.$. This guarantees that for any c of this kind there exists

a unique solution of equation (16) (see e.g. [42]) that we denote by $k(\cdot; k_0, c)$. Recall that $L^1_{loc}(0, +\infty; \mathbb{R})$ is the space of all functions $f : (0, +\infty) \to \mathbb{R}$ that are integrable on all finite intervals. Generally in Economic models one considers less general control strategies, e.g. continuous except for a finite number of points. However, choosing such set of admissible strategies would create technical problems (e.g. non existence of optimal strategies). For this reason we choose to work in this more general setting.

Moreover we assume for simplicity that no borrowing is allowed, so also $k(t) \geq 0$ at any $t \geq 0$. To model the behavior of this consumer we suppose that he/she wants to maximize a certain function U (which should measure the satisfaction of the agent) of the consumption path $c(\cdot)$ up to the time horizon $T \in [0, +\infty]$. Usually such function U (called the intertemporal discounted utility from consumption) is given by

$$U(k; c) = \int_0^T e^{-\rho t} u(c(t))\, dt + e^{-\rho T} \phi(k(T)), \qquad \text{for } T < +\infty$$

$$U(c) = \int_0^{+\infty} e^{-\rho t} u(c(t))\, dt, \qquad \text{for } T = +\infty$$

where the function $\rho > 0$ is th so-called discount rate of the agent, while $u : [0, +\infty) \mapsto \mathbb{R}^+$ is the instantaneous utility from consumption and $\phi : [0, +\infty) \mapsto \mathbb{R}^+$ is the utility from remaining capital stock. We recall that the functions u and ϕ are generally choosen strictly increasing, concave (for economic reasons) and two times differentiable (for simplicity reasons). The standard choice of u is the so-called C.E.S. (Constant Elasticity of Substitution) utility function which is given by

$$u_\sigma(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \qquad for \quad \sigma > 0,\ \sigma \neq 1$$

$$u_1(c) = \log c \qquad for \quad \sigma = 1.$$

(we observe that, in the case $\sigma > 0,\ \sigma \neq 1$ one generally drops for simplicity the constant $(1 - \sigma)^{-1}$ in the function u_σ, without changing the problem).

Summarizing, for the case $T = +\infty$, we have the following maximization problem (P): fixed the initial endowment k_0, we maximize the intertemporal discounted utility

$$U_\sigma(c) = \int_0^{+\infty} e^{-\rho t} u_\sigma(c(t))\, dt$$

over all consumption strategies $c \in \mathcal{A}(k_0)$ where

$$\mathcal{A}(k_0) = \left\{ c \in L^1_{loc}(0, +\infty; \mathbb{R}) :\ c \geq 0\ a.e.;\quad k(\cdot; k_0, c) \geq 0\ a.e. \right\}$$

(i.e. c admissible starting at k_0).

Remark 5.1 *This problem has also an important meaning as a macroeconomic model where the agent is a representative agent of a certain community (see [6]). In this case the equation for the capital stock (i.e. the "state" equation) is substituted by the more general one*

$$\dot{k}(t) = F(k(t)) - c(t),\ t \in [0, T]; \qquad k(0) = k_0. \tag{17}$$

where $F(k)$ stands for the production function of the model (generally linear or concave).

5.2 Example 2: optimal investment

Consider a firm that produces goods using a certain amount of capital stock k (i.e. the machines used for production or the cattle). A first model of the evolution of the stock k can be the following

$$\dot{k}(t) = I(t) - \mu k(t), \quad k(0) = k_0, \tag{18}$$

where μ is the decay rate of capital (i.e. the machines become older and can broke) while I is the investment rate. The firm can choose the investment strategy respecting some constraints. For example we assume that $I \in L^1_{loc}(0, +\infty; \mathbb{R})$ and that, calling $k(\cdot; k_0, I)$ the corresponding unique solution of (18) we have $k(t; k_0, I) \geq 0$ for every $t \geq 0$ (note that we allow for negative investments). We model the behavior of the firm assuming that it wants to maximize the discounted profit Π over a fixed time horizon $T \in [0, +\infty]$

$$\Pi(k; I) = \int_0^T e^{-rt} f_0(k(t), I(t))dt + e^{-rT} f_1(k(T)), \qquad \text{for } T < +\infty$$

$$\Pi(k; I) = \int_0^{+\infty} e^{-rt} f_0(k(t), I(t))dt, \qquad \text{for } T = +\infty$$

where $r > 0$ is the interest rate (that we assume to be constant for simplicity), $f_0(k, I)$ gives the istantaneous profit rate for given levels of capital stock and investment rate and $f_1(k)$ gives the profit for keeping a quantity of capital k at the end of the period (e.g. the market value of it).

A typical example of function f_0 is

$$f_0(k, I) = f_{01}(k) - f_{02}(I)$$

where $f_{01} \in C^2([0, +\infty))$ is strictly increasing and concave and $f_{02} \in C^2(\mathbb{R})$ is strictly concave and superlinear (i.e. $\lim_{I \to +\infty} f_{02}(I)/|I| = +\infty$). This is the classical optimal investment problem with adjustment costs:

$$\max \Pi(k; I) = \max \int_0^{+\infty} e^{-\rho t}[f_{01}(k(t)) - f_{02}(I(t))]dt,$$

$$\dot{k}(t) = I(t) - \mu k(t), \quad k(0) = k_0,$$

subject to the usual constraint $k \geq 0$ (for a discussion see e.g. [40, Ch. 8.E]).

6 Dynamic programming

The departure point of dynamic programming (DP) method is the idea of embedding a given optimal control problem (OCP) into a family of OCP indexed by the initial data (t_0, x_0). This means that we keep the horizon T fixed and we let the data (t_0, x_0) vary[10].

[10] In $([0, T+1] \cap \mathbb{N}) \times X$ in the discrete time case and in in $([0, T] \cap \mathbb{R}) \times X$ in the continuous time case.

and look at the relationship between such family of problems in the same spirit of the so-called *envelope theorem* in static optimization.

The core of such relationship can be summarized in the following sentence: "The second part of an optimal trajectory is still optimal". This has probably been the first statement of the so-called Bellman's Optimality Principle, formulated in the '50's, which is the departure point of DP. For an history of dynamic programming method one can see e.g. [43] or [4], [11]. Here we recall the main ideas and statements.

The main idea of DP is the following. First state precisely the relationship between problems with different data (the Bellman's Optimality Principle). Then use these relationship (eventually in a modified form: this happens especially in the continuous time case where the infinitesimal form is studied) to get information about optimal control strategies. The key tool to find these relationship is the value function of the problem, see Definition 3.10. For a detailed discussion of DP with proofs see e.g , [11] for the discrete time case. and [42], [43] for the continuous time case.

Before to pass to precise statements of theorems we give an outline of the main ideas of dynamic programming. The list is purely a rough indication.

1. Define the value function of the problem as in Definition 3.10: this is a function of the initial time and the initial state (t_0, x_0).
2. Find a functional equation for V, the so-called Bellman equation, which is always satisfied by V (Theorem 1).
3. Find a solution v, if possible, of the Bellman equation and prove that such solution is the value function V.
4. Characterize optimal solution in terms of the value function (Theorem 6.4).
5. Use this characterization to find a feedback optimal map and so, via the closed loop equation, the optimal couples.

Practically, of course, step 1 is just a definition, step 2 is an application of Bellman's principle which very often satisfied. Step 4 is a direct application of known theorems, so the job there is to check if suitable hypotheses are verified. Steps 3 and 5 involve computations that are some times easy, sometimes simply impossible.

6.1 The discrete time case

Let us start from the following

Theorem 1 (Bellman equation). *Let Hypothesis 3.12 hold for problem (P). Then for every $(t_0, x_0) \in ([0,T] \cap \mathbb{N}) \times X$ and $\tau \in [t_0, T]$ we have*

$$V(t_0, x_0) = \sup_{c \in \mathcal{C}_{ad}(t_0, x_0)} \left\{ \sum_{s=t_0}^{\tau} f_0\left(s, x\left(s; t_0, x_0, c(\cdot)\right), c(s)\right) + V\left(\tau + 1, x\left(\tau + 1; t_0, x_0, c(\cdot)\right)\right) \right\} \tag{19}$$

or, equivalently,

$$\begin{aligned} V(t_0,x_0) &= \sup_{c(\cdot)\in\mathcal{C}_{ad}(t_0,x_0)} \{f_0(t_0,x_0,c(t_0)) + V(t_0+1, x(t_0+1;t_0,x_0,c(\cdot)))\} = (20) \\ &\sup_{c(\cdot)\in\mathcal{C}_{ad}(t_0,x_0)} \{f_0(t_0,x_0,c(t)) + V(t_0+1, f_D(t_0,x_0,c(t_0)))\} \\ &\sup_{c\in C(t_0,x_0)} \{f_0(t_0,x_0,c) + V(t_0+1, f_D(t_0,x_0,c))\} \end{aligned}$$

where we have set

$$C(t_0,x_0) = \{c\in C: \quad \exists c(\cdot)\in\mathcal{C}_{ad}(t_0,x_0) \ \ with\ c(t_0)=c\}$$

Proof. We provide the proof of (20) in the finite horizon case. The others are similar. By definition the value function at a given point (t_0,x_0) is

$$V(t_0,x_0) \overset{def}{=} \sup_{c\in\mathcal{C}_{ad}(t_0,x_0)} \left\{ \sum_{t=t_0}^{T} f_0(t,x(t),c(t)) + \phi_0(x(T+1)) \right\}$$

then, by splitting the sum

$$V(t_0,x_0) = \sup_{c\in\mathcal{C}_{ad}(t_0,x_0)} \left\{ f_0(t_0,x_0,c(t_0)) + \sum_{t=t_0+1}^{T} f_0(t,x(t),c(t)) + \phi_0(x(T+1)) \right\}.$$

Now observe that, for any $c(\cdot)\in\mathcal{C}_{ad}(t_0,x_0)$, the strategy $c(\cdot)|_{[t_0+1,T]\cap\mathbb{N}}$ (i.e. $c(\cdot)$ restricted over the times after t_0) belongs to $\mathcal{C}_{ad}(t_0+1,x(t_0+1))$ (where $x(t_0+1) = x(t_0+1;t_0,x_0,c(\cdot)) = f(t_0,x_0,c(t_0))$) so that we have

$$\begin{aligned} &\sum_{t=t_0+1}^{T} f_0(t,x(t),c(t)) + \phi_0(x(T+1)) \\ \leq &\sup_{c\in\mathcal{C}_{ad}(t_0+1,x(t_0+1))} \sum_{t=t_0+1}^{T} f_0(t,x(t),c(t)) + \phi_0(x(T+1)) = V(t_0+1,x(t_0+1)) \end{aligned}$$

which means

$$V(t_0,x_0) \leq \sup_{c(\cdot)\in\mathcal{C}_{ad}(t_0,x_0)} \{f_0(t_0,x_0,c(t_0)) + V(t_0+1, f_D(t_0,x_0,c(t_0)))\}.$$

The proof of the opposite inequality is left as an exercise. For the proof in the infinite horizon case see e.g. [11]. ■

Remark 6.1 *The above theorem provides a functional equation*[11] *for the value function V. The treatment of it is much different depending on the finiteness of the horizon T of the problem. In fact in the finite horizon case we have that*

[11] By functional equation we mean an equation where the unknown is a function.

- *The value function V is the unique solution of (19).*
- *The value function can be calculated using a backward recursive algorithm given by (20). In fact we have, by the definition of value function*

$$V(T+1, x_0) = \phi_0(x_0)$$

and, taking (20) for $t_0 = T$

$$\begin{aligned} V(T, x_0) &= \sup_{c \in C(T, x_0)} \{f_0(T, x_0, c) + V(T+1, f_D(T, x_0, c))\} \\ &= \sup_{c \in C(T, x_0)} \{f_0(T, x_0, c) + \phi_0(f_D(T, x_0, c))\}. \end{aligned}$$

This allows to calculate $v(T, x_0)$. Then, using again (20) for $t_0 = T-1$

$$V(T-1, x_0) = \sup_{c \in C(T-1, x_0)} \{f_0(T-1, x_0, c) + V(T, f_D(T-1, x_0, c))\}.$$

and so, recursively, we can calculate $V(t_0, x_0)$ for every $0 \leq t_0 < T$. Observe that substantially we are dividing the problem of optimizing over $T - t_0 + 1$ variables $c(t_0), \ldots, c(T)$, in $T - t_0 + 1$ "static" parametric problems taken backward sequentially in time. To write explicitly this fact choose a case where $T = 1$ and note that

$$V(0, x_0) = \sup_{c(\cdot) \in \mathcal{C}_{ad}(0, x_0)} f_0(0, x_0, c(0)) + f_0(1, x(1), c(1)) + \phi_0(2, x(2))$$

$$= \sup_{c_0 \in C(0, x_0)} \left\{ f_0(0, x_0, c_0) + \sup_{c_1 \in C(1, x(1))} \{f_0(1, f_D(0, x_0, c_0), c_1) + \phi_0(2, f(1, f_D(0, x_0, c_0), c_1))\} \right\}$$

Remark 6.2 *In the infinite horizon case it is not true in general that the value function is the unique solution to the Bellman equation. So given a solution w to it we need to understand if it is or not the value function. A sufficient condition for this is e.g. that*

$$\lim_{t \to +\infty} w(t, x(t)) = 0$$

on every admissible trajectory $x(\cdot)$.

Before to give the next result we recall that, if c^* is an optimal strategy at (t_0, x_0), the associate state trajectory at time t is denoted by $x^*(t) = x(t; t_0, x_0, c^*(\cdot))$. The couple (x^*, c^*) is an *optimal state-control couple.*

Theorem 6.3 (Bellman's optimality principle). *Let Hypotheses 3.12 hold for problem (P). Then for every $(t_0, x_0) \in ([0, T] \cap \mathbb{N}) \times X$ and $t \in [t_0, T] \cap \mathbb{N}$ we have:*

$$c^* \text{ optimal at } (t_0, x_0) \Longrightarrow c^*|_{[t, T] \cap \mathbb{N}} \text{ optimal at } (t, x^*(t)).$$

Roughly speaking: "Every second part of an optimal trajectory is optimal".

Theorem 6.4 (Optimality conditions via dynamic programming). *Let Hypotheses 3.12 hold for problem (P). Let $T < +\infty$. Then for every $(t_0, x_0) \in ([0,T] \cap \mathbb{N}) \times X$ we have:*

$$\begin{array}{c} c^*(\cdot) \text{ optimal at } (t_0, x_0) \\ \Updownarrow \\ c^*(t) \in \arg\max_{c \in C(t,x^*(t))} \{f_0(t, x^*(t), c) + V(t+1, f(t, x^*(t), c))\}, \qquad \forall t \in [t_0, T] \cap \mathbb{N} \end{array} \tag{21}$$

Let now $T = +\infty$. Then

$$\begin{array}{c} c^*(\cdot) \text{ optimal at } (t_0, x_0) \\ \Updownarrow \\ c^*(t) \in \arg\max_{c \in C(t,x^*(t))} \{f_0(t, x^*(t), c) + V(t+1, f(t, x^*(t), c))\}, \qquad \forall t \in \mathcal{T} \\ \text{and} \qquad \lim_{t \to +\infty} V(t, x^*(t)) = 0 \end{array}$$

Remark 6.5 *Equivalently we may write (21) as*

$$\begin{array}{c} c^*(\cdot) \text{ optimal at } (t_0, x_0) \\ \Updownarrow \end{array}$$

$$\begin{aligned} & f_0(t, x^*(t), c^*(t)) + v(t+1, f(t, x^*(t), c^*(t))) \\ = & \max_{c \in C(t,x^*(t))} \{f_0(t, x^*(t), c) + v(t+1, f(t, x^*(t), c))\}, \qquad \forall t \in \mathcal{T}. \end{aligned} \tag{22}$$

Moreover, defining the (possibly) multivalued function $G : \mathcal{T} \times X \to C$ as, for every $(t,x) \in \mathcal{T} \times X$,

$$G(t,x) = \arg\max_{c \in C(t,x)} \{f_0(t,x,c) + v(t+1, f(t,x,c))\}$$

the above optimality condition (22) reads as

$$c^*(t) \in G(t, x^*(t)) \qquad \forall t \in \mathcal{T}. \tag{23}$$

This is exactly what we called an Optimal Feedback map in Section 3.2.3.

Remark 6.6 *Note that, once we calculated the value function (if possible) solving the Bellman equation, the above Theorem 6.4 and the subsequent remark tell us a way to calculate the optimal couples. It is enough to use (23) putting it into the state equation obtaining the closed loop equation (see Section 3.2.3)*

$$\begin{cases} x^*(t+1) \in f(t, x^*(t), G(t, x^*(t))), & \forall t \in \mathcal{T} \\ x(t_0) = x_0 \end{cases} \tag{24}$$

Solving, if possible, this difference equation (inclusion) we find the optimal trajectory $x^*(\cdot)$. The optimal strategy at time t is then given by substituting the $x^*(t)$ into (23). In our cases we will always have G monovalued, so a veritable function.

Remark 6.7 *In some infinite horizon cases we may be able to find a solution of the Bellman equation but we may not be able to prove that such solution is the value function (e.g. if we are not able to exploit the sufficient condition of Remark 6.2). In such case we may use the following variant of Theorem 6.4:*

Assume that w solves the Bellman equation and fix $t_0 \in [0,T] \cap \mathbb{N}$, $x_0 \in X$. Assume that we can find an admissible couple $(\bar{c}(\cdot), \bar{x}(\cdot)) \in \mathcal{C}(t_0, x_0)$ such that

$$\begin{aligned} & f_0\left(t, \bar{x}\left(t\right), \bar{c}\left(t\right)\right) + w\left(t+1, f\left(t, \bar{x}\left(t\right), \bar{c}\left(t\right)\right)\right) \\ = & \max_{c \in C(t, \bar{x}(t))} \left\{ f_0\left(t, \bar{x}\left(t\right), c\right) + w\left(t+1, f\left(t, \bar{x}\left(t\right), c\right)\right)\right\}, \qquad \forall t \in \mathcal{T}. \end{aligned} \tag{25}$$

and

$$\text{and} \qquad \lim_{t \to +\infty} w\left(t, \bar{x}\left(t\right)\right) = 0$$

then $w(t_0, x_0) = V(t_0, x_0)$ and the couple $(\bar{c}(\cdot), \bar{x}(\cdot))$ is optimal at (t_0, x_0).

6.1.1 The infinite horizon case with discount

We consider here the infinite horizon discrete time problem where $f_0 = \beta^t f_1$ and f, g are autonomous, so

(P_1) **Maximize the functional**

$$J\left(t_0, x_0, c\left(\cdot\right)\right) = \sum_{t=t_0}^{+\infty} \beta^t f_1\left(x\left(t\right), c\left(t\right)\right) dt$$

with state equation

$$\begin{cases} x(t+1) = f(x(t), c(t)); & t \in \mathcal{T} = [t_0, +\infty) \\ x(t_0) = x_0 & x_0 \in X \subseteq \mathbb{R}^n, \end{cases}$$

and constraints

$$x\left(t\right) \in X, \quad c\left(t\right) \in C, \quad \forall t \in \mathcal{T}$$

$$g\left(x\left(t\right), c\left(t\right)\right) \leq 0, \quad \forall t \in \mathcal{T}.$$

As seen in Remark 3.14 we have that, for every $(t_0, x_0) \in \mathcal{T} \times X$

$$V\left(t_0, x_0\right) = \beta^{t_0} V\left(0, x_0\right).$$

So in this case it is enough to know the function $V_0\left(x_0\right) = V\left(0, x_0\right)$ to calculate the value function. This means that we can avoid to let the initial time vary, so we take $t_0 = 0$ and consider the value function

$$V_0\left(x_0\right) = \sup_{c(\cdot) \in \mathcal{C}_{\text{ad}}(0, x_0)} \sum_{t=0}^{+\infty} \beta^t f_1\left(x\left(t\right), c\left(t\right)\right) = \sup_{c(\cdot) \in \mathcal{C}_{\text{ad}}(0, x_0)} J\left(0, x_0, c\left(\cdot\right)\right)$$

Assume that

Hypothesis 6.8 *For every $x_0 \in X$ we have*

$$\mathcal{C}_{\text{ad}}(0, x_0) \neq \emptyset$$

and

$$V_0(x_0) < +\infty$$

Remark 6.9 *Note that we accept that $V_0(x_0) = -\infty$ at some $x_0 \in X$.*

In this case the above Theorems 1, 6.3, 6.4 are rewritten as follows

Theorem 2 (Bellman equation). *Let Hypothesis 6.8 hold for problem (P_1). Then setting for everey $x_0 \in X$*

$$C(x_0) = \{c \in C : \quad \exists c(\cdot) \in \mathcal{C}_{ad}(0, x_0) \ \text{ with } c(0) = c\},$$

we have

$$V_0(x_0) = \sup_{y \in C(x_0)} \{f_1(x_0, c) + \beta V_0(f(x_0, c))\} \tag{26}$$

or, equivalently, for every $x_0 \in X$ and $\tau \in \mathcal{T}$,

$$V_0(x_0) = \sup_{c(\cdot) \in \mathcal{C}_{\text{ad}}(0, x_0)} \left\{ \sum_{t=0}^{\tau} f_1(x(t), c(t)) + \beta^{\tau+1} V_0(x(\tau + 1; 0, x_0, c(\cdot))) \right\}. \tag{27}$$

Remark 6.10 *As recalled in Remark 6.2, in the infinite horizon case it is not true in general that the value function is the unique solution to the Bellman equation. So given a solution z to it we need to understand if it is or not the value function. A sufficient condition for this is given in Remark 6.2 and, in the discounted case it says that*

$$\lim_{t \to +\infty} \beta^t z(x(t)) = 0$$

on every admissible trajectory $x(\cdot)$. Another sufficient condition that we will use is that:

- *for every admissible trajectory $x(\cdot)$*

$$\lim_{t \to +\infty} \inf \beta^t z(x(t)) \leq 0$$

- *for every admissible couple $(x(\cdot), c(\cdot))$ there exists another admissible trajectory $(\bar{x}(\cdot), \bar{c}(\cdot))$ with*

$$J(0, x_0, \bar{c}(\cdot)) \geq J(0, x_0, c(\cdot))$$

and

$$\lim_{t \to +\infty} \sup \beta^t z(\bar{x}(t)) \geq 0$$

see on this [29, Propositions 2.7, 2.8].

Theorem 6.11 (Bellman's optimality principle). *Let Hypothesis 6.8 hold for problem* (P_1)*. Then for every* $x_0 \in X$ *and* $t \geq 0$*, we have:*

$$(x^*(\cdot), c^*(\cdot)) \text{ optimal couple at } x_0 \Longrightarrow (x^*(t+\cdot), c^*(t+\cdot)) \text{ optimal couple at } x^*(t).$$

Roughly speaking: "Every second part of an optimal trajectory is optimal".

Theorem 6.12 (Optimality conditions via dynamic programming). *Let Hypothesis 6.8 hold for problem* (P_1)*. Let* $T = +\infty$ *and fix* $x_0 \in X$*. Then we have:*

$$\begin{gathered}
(x^*(\cdot), c^*(\cdot)) \text{ optimal at } x_0 \\
\Updownarrow \\
c^*(t) \in \arg\max_{c \in C(x^*(t))} \{f_1(x^*(t), c) + \beta V_0(f(x^*(t), c))\}, \qquad \forall t \in \mathcal{T} \\
\text{and} \qquad \lim_{t \to +\infty} \beta^t V_0(x^*(t)) = 0
\end{gathered} \tag{28}$$

Remark 6.13 *Observe first that by definition of* $\arg\max$

$$\begin{gathered}
c^*(t) \in \arg\max_{c \in C(x^*(t))} \{f_1(x^*(t), c) + \beta V_0(f(x^*(t), c))\}, \qquad \forall t \in \mathcal{T} \\
\Updownarrow \\
f_1(x^*(t), c^*(t)) + \beta V_0(f(x^*(t), c^*(t))) = \max_{c \in C(x^*(t))} \{f_1(x^*(t), c) + \beta V_0(f(x^*(t), c))\}, \qquad \forall t \in \mathcal{T}.
\end{gathered}$$

Defining a (possibly multivalued) function $G : X \to X$ *as*

$$G(x) = \arg\max_{c \in C(x)} \{f_1(x, c) + \beta V_0(f(x, c))\}$$

we can rewrite the above (28) as

$$\begin{gathered}
(x^*(\cdot), c^*(\cdot)) \text{ optimal at } x_0 \\
\Updownarrow \\
x^*(t+1) \in f(x^*(t), G(x^*(t))), \qquad \forall t \in \mathcal{T} \qquad (29) \\
\text{and} \qquad \lim_{t \to +\infty} \beta^t V_0(x^*(t)) = 0 \qquad (30)
\end{gathered}$$

so to find the optimal trajectories we have to solve a difference equation (possibly a difference inclusion if G *is multivalued) with initial condition* $x(0) = x_0$*. If the solution (or a solution if there are more than one) satisfies (30) then it is an optimal trajectory. The difference equation (inclusion)*

$$\begin{cases} x^*(t+1) \in f(x^*(t), G(x^*(t))), & \forall t \in \mathcal{T} \\ x(0) = x_0 \end{cases}$$

plays the role of the closed loop equation (24) in this case, so it will be also called the closed loop equation for our problem.

We finally observe that still Remark 6.7 can be applied here.

Remark 6.14 *We outline below the main steps to follows to find (if possible) an optimal trajectory for problem* (P_1).

1. *Write the problem in the standard formulation (if it is not already written this way), see on this Exercise 4.*
2. *Write the Bellman equation.*
3. *Find a solution to the Bellman equation.*
4. *Prove that such solution is the value function using Remark 6.10.*
5. *Use Remark 6.13 to find optimal strategies, namely:*
   (a) *find the feedback map* G;
   (b) *solve the difference equation (inclusion) (29) with the initial condition* $x(0) = x_0$;
   (c) *check if the solution (or a solution if there are many) satisfies (30).*

Remark 6.15 *If in some point* f_1 *is not finite (see next exercise at point 3 or [29, Section 3.1.1]) everything can be studied without problems. Usually one includes also points where* f_1 *is* $-\infty$ *in* X.

6.2 Exercises (DP-discrete time case)

Exercise 1 *(Cake eating, finite horizon). Consider the problem, for fixed* $T \in [0, +\infty)$ $(\mathcal{T} = [0, T] \cap \mathbb{N})$

$$Maximize \qquad U_1(c(\cdot)) = \sum_{t=0}^{T} \beta^t U_0(c(t))$$

with the constraints :

$$\begin{array}{rcll} c(t) & \geq & 0 \quad \forall t \in \mathcal{T}, & \text{(control constraint)}, \\ x(t) & \geq & 0, \quad \forall t \in \mathcal{T}, & \text{(state constraint)}, \end{array}$$

$$\begin{cases} x(t+1) = x(t) - c(t) & \forall t \in \mathcal{T}, \\ x(0) = x_0, \end{cases} \qquad \text{(state equation)}.$$

1. *Rewrite it in the (CV) form by eliminating the variable* c.
2. *Solve the problem in the case* $T = 1$, $U_0(c) = \frac{c^a}{\alpha}$ $(\alpha \in (0, 1])$ *by using static optimization tecniques.*
3. *Solve the problem in the case* $T = 1$, $U_0(c) = \frac{c^a}{\alpha}$ $(\alpha \in (0, 1])$ *by using DP.*

Solution. The first point is contained in Section 3.1.5. The others are easy applications. We outline them below.

We ned to maximize the function of the two variables $(c(0), c(1))$

$$U_1(c(0), c(1)) = U_0(c(0)) + \beta U_0(c(1))$$

under the constraints

$$c(0) \geq 0, \qquad c(1) \geq 0, \qquad c(0) + c(1) \leq x_0$$

The region is a triangle. Using static optimization techniques we see that:

- the objective function is strictly concave and continuous and defined on a convex closed region. This implies that there exists a unique maximum point.
- there are no interior critical point so the maximum point lies in the boundary.
- the maximum point must live on the segment

$$c(0) \geq 0, \qquad c(1) \geq 0, \qquad c(0) + c(1) = x_0.$$

Using DP we see that

$$c(1) = x(1) = x_0 - c(0)$$

so that $c(0)$ solves the static optimization problem

$$\max_{c(0)\in[0,x_0]} U_0(c(0)) + \beta U_0(x_0 - c(0)).$$

We live calculations as exercise. Please check that the two methods give the same solution

Exercise 2 *(Cake eating, infinite horizon). Consider the problem, for fixed $T = +\infty$ ($\mathcal{T} = \mathbb{N}$)*

$$Maximize \qquad U_2(x_0, c(\cdot)) = \sum_{t=0}^{+\infty} \beta^t U_0(c(t))$$

with the constraints :

$$\begin{aligned} x(t) &\geq 0, \quad c(t) \geq 0 \quad \forall t \in \mathcal{T}, \\ x(t+1) &= x(t) - c(t) \qquad \forall t \in \mathcal{T}. \\ x(0) &= x_0 \end{aligned}$$

Here $X = [0, +\infty)$ and $C = [0, +\infty)$

1. *Write the Bellman equation.*
2. *For $U_0(c) = \frac{c^\alpha}{\alpha}$ ($\alpha \in (0,1)$) show that, for suitable value of the parameter A the function*

   $$z(x) = Ax^\alpha$$

   is a solution to the Bellman equation. Prove that $z = V_0$ using Remark 6.10. Find the optimal strategies.

3. *For $U_0(c) = \frac{c^a}{\alpha}$ ($\alpha \in (-\infty, 0)$) show that, for suitable value of the parameter A the function*

$$z(x) = Ax^\alpha$$

is a solution to the Bellman equation. Prove that $z = V_0$ using Remark 6.10. Find the optimal strategies.

4. *For $U_0(c) = \ln c$ show that, for suitable value of the parameters A and B the function*

$$z(x) = A + B \ln x$$

is a solution to the Bellman equation. Prove that $z = V_0$ using Remark 6.10. Find the optimal strategies.

5. *For $U_0(c) = c$ show that, for suitable value of the parameter A the function*

$$z(x) = Ax$$

is a solution to the Bellman equation. Prove that $z = V_0$ using Remark 6.10. Find the optimal strategies.

Solution:

1. Since $f_1(x, y) = U_0(c)$ in this case the Bellman equation is

$$V_0(x) = \sup_{c \in C(x)} \{U_0(c) + \beta V_0(x - c)\}.$$

recalling that here $C(x) = [0, x]$ due to the state and control constraints.

2. Setting $U_0(c) = \frac{c^a}{\alpha}$ ($\alpha \in (0, 1)$) the Bellman equation is

$$V_0(x) = \sup_{c \in C(x)} \{U_0(c) + \beta V_0(x - c)\}.$$

Set now $z(x) = Ax^\alpha$ ($A \in \mathbb{R}$) and check if such a function is a solution of the Bellman equation above. To see this we substitute in place of V_0 the function z and check if we, for some values of A we get equality *for every* $x \in X$.

$$Ax^\alpha = \sup_{c \in [0,x]} \left\{ \frac{c^\alpha}{\alpha} + \beta A (x - c)^\alpha \right\}. \tag{31}$$

We proceed by calculating the sup in the right hand side. Setting

$$h(c) = \frac{c^\alpha}{\alpha} + \beta A (x - c)^\alpha; \qquad h : [0, x] \to \mathbb{R}$$

we can easily check that h is continuous on $[0, x]$, differentiable in $(0, x)$ and

$$h'(y) = c^{\alpha-1} - \beta A \alpha (x - c)^{\alpha-1},$$

$$h''(y) = (\alpha - 1)\, c^{\alpha-2} + \beta A \alpha\, (\alpha - 1)\, (x - c)^{\alpha-2}.$$

Fix now $A > 0$ (this is reasonable since we are looking for the value function which is positive). Since $\alpha < 1$ we have $h''(c) < 0$ on $(0, x)$ so h is strictly concave there. Moreover

$$\begin{aligned} h'(c) &= 0 \\ &\iff c^{\alpha-1} = \beta A \alpha\, (x - c)^{\alpha-1} \\ &\iff c = (\beta A \alpha)^{\frac{1}{\alpha-1}}\, (x - c) \\ &\iff c\left(1 + (\beta A \alpha)^{\frac{1}{\alpha-1}}\right) = (\beta A \alpha)^{\frac{1}{\alpha-1}}\, x \\ &\iff c = \frac{(\beta A \alpha)^{\frac{1}{\alpha-1}}}{1 + (\beta A \alpha)^{\frac{1}{\alpha-1}}} x \in (0, x) \end{aligned}$$

so by the strict concavity the point

$$c_{\max} = \frac{(\beta A \alpha)^{\frac{1}{\alpha-1}}}{1 + (\alpha \beta A)^{\frac{1}{\alpha-1}}} x = G(x) \tag{32}$$

is the unique maximum point of h in $[0, x]$ (note that we have called it $G(x)$ since below it will become the arg max needed to find the optimal strategy, recall Remark 6.13). Now observe that

$$\begin{aligned} h(c_{\max}) &= \frac{c_{\max}^{\alpha}}{\alpha} + \beta A\, (x - c_{\max})^{\alpha} \\ &= \frac{1}{\alpha}\left(\frac{(\beta A \alpha)^{\frac{1}{\alpha-1}}}{1 + (\beta A \alpha)^{\frac{1}{\alpha-1}}}\right)^{\alpha} x^{\alpha} + \beta A \left(\frac{1}{1 + (\beta A \alpha)^{\frac{1}{\alpha-1}}}\right)^{\alpha} x^{\alpha} \\ &= \frac{x^{\alpha}}{\alpha} \frac{1}{\left(1 + (\beta A \alpha)^{\frac{1}{\alpha-1}}\right)^{\alpha}} \left[(\beta A \alpha)^{\frac{\alpha}{\alpha-1}} + \beta A \alpha\right] \\ &= x^{\alpha} \beta A \left(1 + (\alpha \beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} \end{aligned}$$

where the last step follows since

$$\begin{aligned} (\alpha \beta A)^{\frac{\alpha}{\alpha-1}} + \alpha \beta A &= (\alpha \beta A)^{1 + \frac{1}{\alpha-1}} + \alpha \beta A = (\alpha \beta A) \cdot (\alpha \beta A)^{\frac{1}{\alpha-1}} + \alpha \beta A \\ &\quad \alpha \beta A \left(1 + (\alpha \beta A)^{\frac{1}{\alpha-1}}\right). \end{aligned}$$

Putting this result into equation (31) we get

$$A x^{\alpha} = x^{\alpha} \beta A \left(1 + (\alpha \beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} \qquad \forall x \in X.$$

The latter is verified if and only if the coefficients of x^{α} are equal, i.e.

$$\begin{aligned} A &= \beta A \left(1 + (\alpha \beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} \iff \left(1 + (\alpha \beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} = \beta^{-1} \\ &\iff 1 + (\alpha \beta A)^{\frac{1}{\alpha-1}} = \beta^{\frac{1}{\alpha-1}} \iff (\alpha \beta A)^{\frac{1}{\alpha-1}} = \beta^{\frac{1}{\alpha-1}} - 1. \end{aligned}$$

Now we can take the power $\alpha - 1$ but this is possible if and only if

$$\beta^{\frac{1}{\alpha-1}} - 1 > 0.$$

However this condition is always satisfied in this case as $\beta \in (0,1)$. We then get

$$\alpha\beta A = \left(\beta^{\frac{1}{\alpha-1}} - 1\right)^{\alpha-1} \Longleftrightarrow A = \frac{1}{\alpha}\left(\beta^{\frac{1}{\alpha-1}} - 1\right)^{\alpha-1} = \frac{1}{\alpha}\left(1 - \beta^{\frac{1}{1-\alpha}}\right)^{\alpha-1} > 0.$$

We now prove that $z = V_0$ using Remark 6.10. It is enough to show that, for every admissible trajectory $x(\cdot)$ we have

$$\lim_{t\to+\infty} \beta^t A\left[x(t)\right]^\alpha = 0. \tag{33}$$

But this is obvious since the sequence $x(t)$ is always contained in $[0, x_0]$ and so it is bounded and also $[x(t)]^\alpha$ is bounded. Then the product of a bounded sequence for an infinitesimal one must go to 0.

We finally find the optimal strategies. According to Remark 6.14 we have to

(a) find the feedback map G;

(b) solve the difference equation (inclusion) (29) with the initial condition $x(0) = x_0$;

(c) check if the solution (or a solution if there are many) satisfies (30).

The first duty is already done, recall the equation (32) Let us then solve the closed loop equation

$$\begin{aligned} x(t+1) &= x(t) - G(x(t)) \\ &= x(t) - \frac{(\beta A\alpha)^{\frac{1}{\alpha-1}}}{1 + (\alpha\beta A)^{\frac{1}{\alpha-1}}} x(t) \\ &= \frac{1}{1 + (\alpha\beta A)^{\frac{1}{\alpha-1}}} x(t) = \beta^{\frac{1}{1-\alpha}} x(t), \qquad (t \in \mathbb{N}); \qquad x(0) = x_0. \end{aligned}$$

This is clearly a geometric sequence, so the unique solution is

$$x^*(t) = \beta^{\frac{1}{1-\alpha}t} x_0.$$

The associated control is then

$$c^*(t) = G(x^*(t)) = \frac{(\beta A\alpha)^{\frac{1}{\alpha-1}}}{1 + (\alpha\beta A)^{\frac{1}{\alpha-1}}} \beta^{\frac{1}{1-\alpha}t} x_0.$$

It is immediate to check that the couple $(c^*(\cdot), x^*(\cdot))$ is admissible since it is always positive. Finally we check if (30) is satisfied: this writes

$$\lim_{t\to+\infty} \beta^t A\left[x^*(t)\right]^\alpha = 0$$

but this is surely true since, by (33) we know this is true for every admissible strategy, and also for x^*. So x^* is optimal. It is also the unique optimal trajectory since any other optimal trajectory should satisfy the same optimality conditions that have unique solution in this case.

3. $U_0(c) = \frac{c^a}{\alpha}$ $(\alpha \in (-\infty, 0))$ show that, for suitable value of the parameter A the function
$$z(x) = Ax^\alpha$$
is a solution to the Bellman equation. Prove that $z = V_0$ using Remark 6.10. Find the optimal strategies.

Setting $U_0(c) = \frac{c^a}{\alpha}$ $(\alpha \in (-\infty, 0))$ the Bellman equation is
$$V_0(x) = \sup_{c \in C(x)} \{U_0(c) + \beta V_0(x - c)\}.$$

Set now $z(x) = Ax^\alpha$ $(A \in \mathbb{R})$ and check if such a function is a solution of the Bellman equation above. To see this we substitute in place of V_0 the function z and check if we, for some values of A we get equality *for every* $x \in X$.
$$Ax^\alpha = \sup_{c \in [0,x]} \left\{ \frac{c^\alpha}{\alpha} + \beta A (x - c)^\alpha \right\}.$$

We proceed by calculating the sup in the right hand side. Setting
$$h(c) = \frac{c^\alpha}{\alpha} + \beta A (x - c)^\alpha; \qquad h : [0, x] \to \mathbb{R}$$
we can easily check that h is continuous on $(0, x)$, differentiable in $(0, x)$ and goes to $-\infty$ at both extrema 0 and x. By the Weierstrass Theorem with coercivity h admits a maximum. We have
$$h'(c) = c^{\alpha-1} - \beta A\alpha (x - c)^{\alpha-1},$$
$$h''(c) = (\alpha - 1) c^{\alpha-2} + \beta A\alpha (\alpha - 1) (x - c)^{\alpha-2}.$$
Fix now $A < 0$ (this is reasonable since we are looking for the value function which is negative). Since $\alpha < 0$ we have $h''(c) < 0$ on $(0, x)$ so h is strictly concave there. Moreover, being $\alpha\beta A > 0$ we can write
$$\begin{aligned} h'(c) &= 0 \\ &\iff c^{\alpha-1} = \beta A\alpha (x - c)^{\alpha-1} \\ &\iff c = (\beta A\alpha)^{\frac{1}{\alpha-1}} (x - c) \\ &\iff c \left(1 + (\beta A\alpha)^{\frac{1}{\alpha-1}}\right) = (\beta A\alpha)^{\frac{1}{\alpha-1}} x \\ &\iff c = \frac{(\beta A\alpha)^{\frac{1}{\alpha-1}}}{1 + (\beta A\alpha)^{\frac{1}{\alpha-1}}} x \in (0, x) \end{aligned}$$

so by the strict concavity the point

$$c_{\max} = \frac{(\beta A\alpha)^{\frac{1}{\alpha-1}}}{1+(\alpha\beta A)^{\frac{1}{\alpha-1}}} x = G(x)$$

is the unique maximum point of h in $[0,x]$ (note that we have called it $G(x)$ since below it will become the arg max needed to find the optimal strategy, recall Remark 6.13). Now observe that

$$\begin{aligned} h(c_{\max}) &= \frac{c_{\max}^{\alpha}}{\alpha} + \beta A (x - c_{\max})^{\alpha} \\ &= \frac{1}{\alpha}\left(\frac{(\beta A\alpha)^{\frac{1}{\alpha-1}}}{1+(\beta A\alpha)^{\frac{1}{\alpha-1}}}\right)^{\alpha} x^{\alpha} + \beta A \left(\frac{1}{1+(\beta A\alpha)^{\frac{1}{\alpha-1}}}\right)^{\alpha} x^{\alpha} \\ &= \frac{x^{\alpha}}{\alpha} \frac{1}{\left(1+(\beta A\alpha)^{\frac{1}{\alpha-1}}\right)^{\alpha}} \left[(\beta A\alpha)^{\frac{\alpha}{\alpha-1}} + \beta A\alpha\right] \\ &= x^{\alpha}\beta A \left(1+(\alpha\beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} \end{aligned}$$

where the last step follows since

$$\begin{aligned} (\alpha\beta A)^{\frac{\alpha}{\alpha-1}} + \alpha\beta A &= (\alpha\beta A)^{1+\frac{1}{\alpha-1}} + \alpha\beta A = (\alpha\beta A)\cdot(\alpha\beta A)^{\frac{1}{\alpha-1}} + \alpha\beta A \\ &\quad \alpha\beta A\left(1+(\alpha\beta A)^{\frac{1}{\alpha-1}}\right). \end{aligned}$$

Putting this result into equation (31) we get

$$Ax^{\alpha} = x^{\alpha}\beta A\left(1+(\alpha\beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} \qquad \forall x \in X.$$

The latter is verified if and only if the coefficients of x^{α} are equal, i.e.

$$\begin{aligned} A &= \beta A\left(1+(\alpha\beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} \iff \left(1+(\alpha\beta A)^{\frac{1}{\alpha-1}}\right)^{1-\alpha} = \beta^{-1} \\ &\iff 1+(\alpha\beta A)^{\frac{1}{\alpha-1}} = \beta^{\frac{1}{\alpha-1}} \iff (\alpha\beta A)^{\frac{1}{\alpha-1}} = \beta^{\frac{1}{\alpha-1}} - 1. \end{aligned}$$

Now we can take tha power $\alpha - 1$ but this is possible if and only if

$$\beta^{\frac{1}{\alpha-1}} - 1 > 0.$$

This condition is equivalent to ask $\beta \in (0,1)$ and turns out to be necessary and sufficient fro the existence of the optimal trajectory. Under this condition we get

$$\alpha\beta A = \left(\beta^{\frac{1}{\alpha-1}} - 1\right)^{\alpha-1} \iff A = \frac{1}{\alpha}\left(\beta^{\frac{1}{\alpha-1}} - 1\right)^{\alpha-1} = \frac{1}{\alpha}\left(1-\beta^{\frac{1}{1-\alpha}}\right)^{\alpha-1} < 0.$$

We now prove that $z = V_0$ using Remark 6.10. It is enough to show that, for every admissible trajectory $x(\cdot)$ we have

$$\lim_{t\to+\infty} \beta^t A[x(t)]^{\alpha} = 0.$$

But this is not true since for example the sequence $x(t) = x_0\beta^{-\frac{t}{\alpha}}$ is admissible (please check it!) and gives

$$\lim_{t\to+\infty} \beta^t A\left[x_0\beta^{-\frac{t}{\alpha}}\right]^\alpha = \lim_{t\to+\infty} \beta^t A x_0^\alpha \beta^{-t} = A x_0^\alpha < 0.$$

So one needs to use a more refined argument, the second one of the Remark 6.10. Since or every admissible trajectory $x(\cdot)$ we have

$$\beta^t A\left[x(t)\right]^\alpha \leq 0$$

this implies

$$\lim_{t\to+\infty} \inf \beta^t A\left[x(t)\right]^\alpha \leq 0.$$

Moreover if the series

$$\sum_{t=0}^{+\infty} \beta^t \frac{c(t)^\alpha}{\alpha}$$

diverges then it has payoff $-\infty$. Every admissible trajectory such that the above series is convergent has better payoff than it. Then it is enough for us to prove that for any such couple $(x(\cdot), c(\cdot))$ we have

$$\lim_{t\to+\infty} \sup \beta^t A\left[x(t)\right]^\alpha \geq 0.$$

Now if the series above converges we have that the term goes to 0 as $t \to +\infty$. Since $c(t) \leq x(t)$ for every $t \geq 0$, then (recall that $\alpha < 0$)

$$A\beta^t x(t)^\alpha \geq A\beta^t c(t)^\alpha$$

and so

$$A\beta^t x(t)^\alpha \geq \left(\beta^{\frac{1}{\alpha-1}} - 1\right)^{\alpha-1} \beta^t \frac{c(t)^\alpha}{\alpha}$$

it follows

$$\lim_{t\to+\infty} \sup \beta^t A\left[x(t)\right]^\alpha \geq \left(\beta^{\frac{1}{\alpha-1}} - 1\right)^{\alpha-1} \lim_{t\to+\infty} \sup \beta^t \frac{c(t)^\alpha}{\alpha} = 0$$

and this gives the claim.

We finally find the optimal strategies. According to Remark 6.14 we have to

(a) find the feedback map G;

(b) solve the difference equation (inclusion) (29) with the initial condition $x(0) = x_0$;

(c) check if the solution (or a solution if there are many) satisfies (30).

The first duty is already done, recall the equation (32) Let us then solve the closed loop equation

$$\begin{aligned} x(t+1) &= x(t) - G(x(t)) \\ &= x(t) - \frac{(\beta A\alpha)^{\frac{1}{\alpha-1}}}{1+(\alpha\beta A)^{\frac{1}{\alpha-1}}} x(t) \\ &= \frac{1}{1+(\alpha\beta A)^{\frac{1}{\alpha-1}}} x(t) = \beta^{\frac{1}{1-\alpha}} x(t), \qquad (t \in \mathbb{N}); \qquad x(0) = x_0. \end{aligned}$$

This is clearly a geometric sequence, so the unique solution is

$$x^*(t) = \beta^{\frac{1}{1-\alpha}t} x_0.$$

Finally we check if (30) is satisfied: this writes

$$\lim_{t\to+\infty} \beta^t A [x^*(t)]^\alpha = 0$$

but this is surely true since, by (33) we know this is true for every admissible strategy with convergent series and for such x^* the series converges (please check it). One can also calculate directly the limit getting

$$\lim_{t\to+\infty} \beta^t A [x^*(t)]^\alpha = \lim_{t\to+\infty} \beta^t A \beta^{\frac{1}{1-\alpha}\alpha t} x_0^\alpha = 0$$

since $\beta^{1+\frac{\alpha}{1-a}} < 1$. So x^* is optimal. It is also the unique optimal trajectory since any other optimal trajectory should satisfy the same optimality conditions that have unique solution in this case.

4. This case is deeply analyzed in [29, Section 3.1.1].

5. This case is analyzed in [29, Sections 3.1.3-3.1.4].

6.3 The continuous time case: HJB equations and feedbacks

Here we consider the problem (P) in the continuous time case, finite of infinite horizon T, as described in subsection 3.2.1. We start by the Bellman's Optimality Principle.

Theorem 6.16 (Bellman Optimality Principle). *Let Hypothesis 3.12 hold for problem (P). Then for every $(t_0, x_0) \in ([0,T] \cap \mathbb{R}) \times X$ and $\tau \in [t_0, T] \cap \mathbb{R}$ we have*

$$V(t_0, x_0) = \sup_{c \in \mathcal{C}_{ad}(t_0,x_0)} \left\{ \int_t^\tau f_0(s, x(s; t_0, x_0, c), c(s))\, ds + V(\tau, x(\tau; t_0, x_0, c)) \right\} \tag{34}$$

Proof. See e.g. [42], [43], [4].

Remark 6.17 *The above result also holds for a more general class of problems. Indeed what is needed to prove such result is the following assumption on the set of admissible strategies.*

Hypothesis 6.18 *The family of admissible control strategies* $\{\mathcal{C}_{ad}(t,x)\}_{(t_0,x_0)\in([0,T]\cap\mathbb{R})\times X}$ *satisfies the following properties:*

1. *For every* $0 \leq t_0 \leq \tau < T$, $x \in X$,

$$c(\cdot) \in \mathcal{C}_{ad}(t_0,x_0) \Rightarrow c(\cdot)|_{[\tau,T]\cap\mathbb{R}} \in \mathcal{C}_{ad}(\tau, x(\cdot;t_0,x_0,c))$$

   (i.e. the second part of an admissible strategy is admissible)

2. *For every* $0 \leq t_0 \leq \tau < T$, $x_0 \in X$,

$$c_1 \in \mathcal{C}_{ad}(t_0,x_0), \quad c_2 \in \mathcal{C}_{ad}(\tau, x(\tau;t_0,x_0,c_1)) \Rightarrow c \in \mathcal{C}_{ad}(t_0,x_0)$$

   for c *defined as*

$$c(s) = \begin{cases} c_1(s) & if \quad s \in [t,\tau] \\ c_2(s) & if \quad s \in [\tau,T]\cap\mathbb{R} \end{cases}$$

   (i.e. the concatenation of two admissible strategies is admissible)

Note that the above hypothesis is satisfied if the set of admissible strategies is of the form given for our problem (P).

Equation (34) is a functional equation satisfied by the value function. This is an alternative representation of V that can be useful to determine its properties or even to calculate it. Of course the functional form of (34) is not easy to handle. It is convenient then to find a differential form of it, i.e. the so called Hamilton-Jacobi-Bellman (HJB) equation.

We state it first in the finite horizon case.

Theorem 6.19 *Let* $T < +\infty$. *Assume that Hypothesis 3.12 holds. Assume further that* f_0 *is uniformly continuous,* ϕ *is continuous and* f_C *satisfies assumptions stated in footnote 3. Assume finally that* $V \in C^1([0,T]\times X)$. *Then* V *is a classical*[12] *solution of the following Partial Differential Equation (PDE):*

$$-V_t(t_0,x_0) = H_{MAX}(t_0,x_0,V_x(t_0,x_0)) \qquad (t_0,x_0) \in [0,T]\times X \tag{35}$$

with the final condition

$$V(T,x_0) = \phi(x_0) \qquad x_0 \in X$$

where the function $H_{MAX} : [0,T]\times X\times\mathbb{R}^n \to \mathbb{R}$ *(the "Maximum value Hamiltonian" or, simply, the "Hamiltonian") is given by:*

$$H_{MAX}(t_0,x_0,p) = \sup_{c\in C}\{H_{CV}(t_0,x_0,p;c)\} \tag{36}$$

where

$$H_{CV}(t_0,x_0,p;c) = \langle f_C(t_0,x_0,c), p\rangle + f_0(t_0,x_0,c) \tag{37}$$

Proof. *See e.g. [42], [43], [4].*

[12] In the sense that all derivatives exist and that the equation is satisfied for every $x \in X$.

Remark 6.20 *The equation (35) usually bear the names of Hamilton and Jacobi because such kind of PDE's were first studied by them in connection with calculus of variations and classical mechanics. Bellman was the first to discover its relationship with control problems, see on this [4], [43]. We will call it Hamilton-Jacobi-Bellman (HJB) equation.*

Remark 6.21 *The function $H_{MAX}(t,x,p)$ is usually called (in the mathematics literature) the Hamiltonian of the problem. However in many cases the Hamiltonian is defined differently. In particular in the economic literature the name Hamiltonian (or "current value Hamiltonian" while the other is the "maximun value Hamiltonian") is often used for the function to be maximized in (36). To avoid misunderstandings we will then use the notation*

$$H_{CV}(t,x,p;c) = \langle f(t,x,c), p\rangle + l(t,x,c)$$

for the current value Hamiltonian and

$$H_{MAX}(t,x,p) = \sup_{c\in C} H_{CV}(t,x,p;c)$$

for the maximum value Hamiltonian.

Remark 6.22 *The key issue of the above result is to give an alternative characterization of the value function in term of the PDE (35). In fact this give a very powerful tool to study properties of V and to calculate it by some numerical analysis (at least in low dimension, see on this [4]). And knowing V one can get important information on the optimal state-control trajectories, as we will see below. However to get a real characterization one need a much more powerful result: here we assumed $V \in C^1([0,T]\times X)$ and we did not get uniqueness. A "good" result should state that the value function V is the unique solution of (35) under general hypothesis on the data. Such kind of result have been a difficult problem for many years (see on this [43]) because the usual definitions of classical or generalized solution did not adapt to PDE of HJB type (see e.g. Benton's book [9] for such approach to HJB equations). The problem was solved in the 80ies with the introduction of the concept of viscosity solutions by Crandall and Lions (see e.g. [4]). With this concept it is possible to state that the value function V is the unique "viscosity" solution of (35) under mild assumptions on the data. And now we can satisfy a curiosity that some of you may have had in reading the HJB equation (35): why do we take the double minus sign?. Here is the point: the concept of viscosity solution is "sign" sensitive, i.e. if a function v is a viscosity solution of the PDE*

$$F(t,x,v_t(t,x), v_x(t,x)) = 0$$

this does not imply that v is also a viscosity solution of the same PDE with opposite sign

$$-F(t,x,v_t(t,x), v_x(t,x)) = 0$$

(see on this [4]). This fact suggests to be careful in saying what is exactly the sign of the HJB equation of the problem (P). It turns out that the right sign is exactly

$$-V_t(t,x) = H_{MAX}(t,x,V_x(t,x)) = 0 \qquad (t,x)\in[0,T]\times X$$

which corresponds to take the same sign of the Bellman optimality principle (34), even if we don't like the initial minus sign: do not change sign when talking about viscosity solutions! Of course, talking about classical solutions, as in Theorem 6.19, we can forget the sign: we kept it for coherence.

The HJB equation has a crucial importance for solving the optimal control problem (P). Before to give the main result on it we prove a fundamental identity in next lemma.

Lemma 6.23 *Assume that the hypotheses of Theorem 6.19 hold for problem (P) and let $v \in C^1([0,T] \times X)$ be a classical solution of (35)[13]. Then the following fundamental identity holds: for every $(t_0, x_0) \in [0,T] \times X$, for every $c(\cdot) \in \mathcal{C}_{ad}(t_0, x_0)$, setting $x(s) = x(s; t_0, x_0, c)$ we have*

$$\begin{aligned} & v(t_0, x_0) - J(t_0, x_0; c(\cdot)) \\ = & \int_t^T \left[H_{MAX}(s, x(s), v_x(s, x(s))) - H_{CV}(s, x(s), v_x(s, x(s)); c(s)) \right] ds \end{aligned} \tag{38}$$

and, in particular, $v(t_0, x_0) \geq V(t_0, x_0)$ for every $(t_0, x_0) \in [0,T] \times X$. Moreover the problem of maximizing J is equivalent to the problem of maximizing

$$\bar{J}(t_0, x_0; c(\cdot)) = -\int_t^T \left[H_{MAX}(s, x(s), v_x(s, x(s))) - H_{CV}(s, x(s), v_x(s, x(s)); c(s)) \right] ds$$

Proof. Let us fix initial data $(t_0, x_0) \in [0,T] \times X$ and control strategy $c \in \mathcal{C}_{ad}(t_0, x_0)$. Then calculate, using that v is a classical solution of (35)

$$\begin{aligned} \frac{d}{ds} v(s, x(s)) &= \frac{\partial}{\partial s} v(s, x(s)) + \left\langle x'(s), \frac{\partial}{\partial x} v(s, x(s)) \right\rangle \\ &= -H_{MAX}(s, x(s), v_x(s, x(s))) + \langle x'(s), v_x(s, x(s)) \rangle \end{aligned}$$

$$\begin{aligned} = & -H_{MAX}(s, x(s), v_x(s, x(s))) + \langle f_C(s, x(s), c), v_x(s, x(s)) \rangle \\ & + f_0(s, x(s), c(s)) - f_0(s, y(s), c(s)) \\ = & -H_{MAX}(s, x(s), v_x(s, x(s))) + H_{CV}(s, x(s), v_x(s, x(s)); c(s)) - f_0(s, x(s), c(s)) \end{aligned}$$

Integrating the above identity on $[t_0, T]$ we then get:

$$\begin{aligned} & v(T, x(T)) - v(t_0, x_0) \\ = & -\int_{t_0}^T \left[H_{MAX}(s, x(s), v_x(s, x(s))) - H_{CV}(s, x(s), v_x(s, x(s)); c(s)) \right] ds \\ & -\int_{t_0}^T f_0(s, x(s), c(s))\, ds \end{aligned}$$

iwhich gives (38) by recalling that $v(T, x(T)) = \phi(y(T))$ and rearranging the terms. ■

[13] This may or may not be the value function.

Theorem 6.24 *Let $v \in C^1([0,T] \times X)$ be a classical solution of (35).*

1. *Fix $(t_0, x_0) \in [0,T] \times X$ and assume that there exists an admissible state-control couple $(\bar{x}(\cdot), \bar{c}(\cdot))$ such that*

$$H_{MAX}(s, \bar{x}(s), v_x(s, \bar{x}(s))) - H_{CV}(s, \bar{x}(s), v_x(s, \bar{x}(s)); \bar{c}(s)) = 0 \qquad s \in [t,T], \quad a.e.$$

i.e.

$$\bar{c}(s) \in \arg\max H_{CV}(s, \bar{x}(s), v_x(s, \bar{x}(s)); \bar{c}) \qquad s \in [t,T], \quad a.e.. \tag{39}$$

Then such couple is optimal at (t_0, x_0) and $v(t_0, x_0) = V(t_0, x_0)$ at such point.

2. *Moreover, if we know from the beginning that $v(t_0, x_0) = V(t_0, x_0)$, then every optimal strategy satisfies equation (39).*

3. *Finally, if, for every $(t_0, x_0, p) \in [0,T] \times X \times \mathbb{R}^n$ the map $c \mapsto H_{CV}(t_0, x_0, p; c)$ admits a unique maximum point $G_0(t_0, x_0, p)$, and the closed loop equation*

$$\begin{aligned} x'(s) &= f_C(s, x(s), G_0(s, x(s), v_x(s, x(s)))) \qquad s \in [t,T], \quad a.e.. \\ x(t_0) &= x_0 \end{aligned}$$

has a unique solution $x^*(\cdot)$, and the control strategy

$$c^*(s) = G_0(s, x^*(s), v_x(s, x^*(s)))$$

is admissible, then $v = V$ on $[0,T] \times X$ and $(x^*(\cdot), c^*(\cdot))$ is an optimal state-control couple.

Proof. All the statements are direct consequence of Lemma 6.23.

Remark 6.25 *Part (iii) of Theorem 6.24 substantially states that the map*

$$(s,y) \mapsto G_0(s, y, v_x(s,y)) \stackrel{def}{=} G(s,y)$$

is the unique optimal feedback map for problem (P). In particular this gives existence and uniqueness of an optimal state-control couple in feedback form ("closed loop").

Remark 6.26 *In Lemma 6.23 and in Theorem 6.24 the function v is not necessarily the value function. Of course, if we know (for example from Theorem 6.19) that the value function V is a classical solution of equation (35) it is natural to choose $v = V$.*

Remark 6.27 *The above results can be used only in few cases, even interesting. Indeed the HJB equation (35) does not admit in general a classical solution. It is possible to give an extended nonsmooth version of the above results by using the concept of viscosity solutions, of course loosing some nice formulation, see on this [4].*

Remark 6.28 *Note that the above results do not need uniqueness of solutions of (35).*

6.4 DP in continuous time: autonomous problem with infinite horizon

As in the discrete time case when we deal with autonomous infinite horizon problems with discount factor, the HJB equation can be simplified. In this section we present and prove this fact for the continuous time case.

Consider the problem $\left(\bar{P}\right)$ of maximizing the functional

$$J\left(t_0, x_0; c(\cdot)\right) = \int_{t_0}^{+\infty} e^{-\lambda s} f_1\left(x\left(s\right), c\left(s\right)\right) ds$$

where $x\left(\cdot\right) = x\left(\cdot; t_0, x_0, c\right)$ is the solution of the state equation

$$\begin{aligned} x'\left(s\right) &= f_C\left(x\left(s\right), c\left(s\right)\right); \quad s \geq t_0 \\ x\left(t_0\right) &= x_0 \in X \end{aligned}$$

and

$$c(\cdot) \in \mathcal{C}_{ad}\left(t_0, x_0\right) \overset{def}{=} \left\{c : [t, +\infty) \mapsto C : \quad c\left(s\right) \in C, \quad x\left(s; t_0, x_0, c\right) \in X \quad \forall s \geq t_0\right\}$$

for given sets $C \subseteq \mathbb{R}^k$ (the control space) and $X \subseteq \mathbb{R}^n$ (the state space). Problem $\left(\bar{P}\right)$ is nothing but problem (P) with infinite horizon, autonomous dynamic f_C and $f_0\left(t, x, u\right) = e^{-\lambda s} f_1\left(x, u\right)$.

The current value Hamiltonian is

$$\begin{aligned} H_{CV}\left(t, x, p; c\right) &= \left\langle f_C\left(t, x, c\right), p\right\rangle + e^{-\lambda t} f_1\left(x, c\right) = e^{-\lambda t}\left[\left\langle f_C\left(x, c\right), e^{\lambda t} p\right\rangle + f_1\left(x, c\right)\right] \\ &= e^{-\lambda t} H_{0CV}\left(x, e^{\lambda t} p; c\right) \end{aligned}$$

where we set $H_{0CV}\left(x, q; c\right) = \left\langle f_C\left(x, c\right), q\right\rangle + f_1\left(x, u\right)$. The maximum value Hamiltonian is

$$H_{MAX}\left(t, x, p\right) = \sup_{c \in C} H_{CV}\left(t, x, p; c\right) = e^{-\lambda t} \sup_{c \in C} H_{0CV}\left(x, e^{\lambda t} p; u\right) \overset{def}{=} e^{-\lambda t} H_{0MAX}\left(x, e^{\lambda t} p\right)$$

Moreover the value function is, for $t_0 \geq 0$ and $x_0 \in X$,

$$V\left(t_0, x_0\right) = \sup_{c(\cdot) \in \mathcal{C}_{ad}(t_0, x_0)} J\left(t_0, x_0; c(\cdot)\right)$$

and we have

$$\begin{aligned} V\left(t_0, x_0\right) &= \sup_{c \in \mathcal{C}_{ad}(t_0, x_0)} \int_{t_0}^{+\infty} e^{-\lambda s} f_1\left(x\left(s\right), c\left(s\right)\right) ds = e^{-\lambda t_0} \sup_{c \in \mathcal{C}_{ad}(t_0, x_0)} \int_{t_0}^{+\infty} e^{-\lambda(s-t)} f_1\left(x\left(s\right), c\left(s\right)\right) ds \\ &= e^{-\lambda t_0} \sup_{c \in \mathcal{C}_{ad}(t_0, x_0)} \int_{0}^{+\infty} e^{-\lambda \tau} f_1\left(x\left(t+\tau\right), c\left(t+\tau\right)\right) d\tau. \end{aligned}$$

Now, being f autonomous we have

$$c(\cdot) \in \mathcal{C}_{ad}\left(t_0, x_0\right) \Leftrightarrow c\left(t_0 + \cdot\right) \in \mathcal{C}_{ad}\left(0, x_0\right)$$

so that

$$V(t_0, x_0) = e^{-\lambda t_0} \sup_{c \in \mathcal{C}_{ad}(0,x_0)} \int_0^{+\infty} e^{-\lambda \tau} f_1(x(s; 0, x_0, c), c(s))\, ds = e^{-\lambda t_0} V(0, x_0).$$

Then, if $V \in C^1([0, +\infty) \times X)$ solves the HJB equation (here the sign does not matter since we are talking of classical solutions, but we keep for coherence the right sign for viscosity)

$$-V_t(t_0, x_0) = H_{MAX}(t_0, x_0, V_x(t_0, x_0)),$$

we have

$$\lambda e^{-\lambda t_0} V(0, x_0) - e^{-\lambda t_0} H_{0MAX}\left(x_0, e^{\lambda t_0}\left(e^{-\lambda t_0} V_x(0, x_0)\right)\right) = 0$$

so the function $V_0(x_0) = V(0, x_0)$ satisfies

$$\lambda V_0(x_0) = H_{0MAX}(x_0, \nabla V_0(x_0)) = 0. \tag{40}$$

In studying the problem $\left(\bar{P}\right)$. It can be then convenient to study the PDE (40) instead of (35). This true especially when $n = 1$, as in our examples, since in this case (40) is just an ODE and V_0 is just a function of one variable. All the results of DP method (in particular Theorem 6.24 remains true with obvious changes (exercise!).

6.5 DP in continuous time: examples

Now we want to apply the DP method to our examples. First we give an outline of the main steps of the DP method in the simplest cases (i.e. when the main assumptions of the above results are verified). We will try to do the following steps.

1. Calculate the Hamiltonians H_{CV} and H together with $\arg\max H_{CV}$.
2. Write the HJB equation and find a classical solution v.
3. Calculate the feedback map G, then solve the closed loop equation finding the optimal state-control couple.

Of course in general it will be impossible to perform such steps. In particular step 2 is generally impossible. However, it can be interesting, in dealing with an economic problem, to study properties of optimal state-control trajectories without knowing them in closed form (i.e. explicitly). To establish such properties it can be enough to know that that the value function V is a solution (classical or viscosity, possibly unique) of HJB equation and that it enjoys some regularity properties (i.e. concavity or differentiability or else) and then infer from them some properties of optimal couples via the closed loop equation (see [18] on this). In any case, in our example we will see a very nice case where everything can be explicitly calculated.

6.6 DP method in the simplest AK model

We develop the calculation in three parts.

6.6.1 Estimates for the value function

The following proposition is useful to establish when the value function is finite.

Proposition 6.29 *Let $a = \rho - r(1-\sigma) > 0$. Then, for any $k_0 \geq 0$ we have, for $\sigma \in (0,1)$ and $c \in \mathcal{A}(k_0)$,*

$$\begin{aligned} 0 &\leq U_\sigma(c) \leq \frac{\rho}{1-\sigma} k_0^{1-\sigma} \int_0^{+\infty} s^\sigma e^{-as}\, ds \\ &= \frac{\rho}{1-\sigma} \frac{\Gamma(1+\sigma)}{a^{1+\sigma}} k_0^{1-\sigma} \end{aligned}$$

while, for $\sigma = 1$

$$U_\sigma(c) \leq \rho \int_0^{+\infty} e^{-\rho s} s \left[rs + \log \frac{k_0}{s} \right] ds$$

and, for $\sigma > 1$

$$0 > U_\sigma(c) \geq \frac{\rho}{1-\sigma} \frac{\Gamma(1+\sigma)}{a^{1+\sigma}} k_0^{1-\sigma}.$$

This result show, in particular, that, when $a > 0$, and $\sigma \in (0,1)$ the intertemporal utility functional $U_\sigma(c)$ is finite and uniformly bounded for every admissible consumption strategy (while for $\sigma \geq 1$ it is only bounded from above). So it gives a bound for the value function.

6.6.2 The Hamiltonians

The current value Hamiltonian H_{0CV} of our problem (P_σ) does not depend on t and is given, for $\sigma \neq 1$ by

$$H_{0CV}(k,p;c) = rkp - cp + \frac{c^{1-\sigma}}{1-\sigma} \qquad k,p \in \mathbb{R}, \quad k \geq 0;$$

$$c \in [0,+\infty), \text{ if } \sigma < 1, \text{ and } c \in (0,+\infty), \text{ if } \sigma > 1$$

and, for $\sigma = 1$ by

$$\begin{aligned} H_{0CV}(k,p;c) &= rkp - cp + \log c \\ k,p &\in \mathbb{R}, \quad t,k \geq 0; \quad c \in (0,+\infty). \end{aligned}$$

Note that it is the sum of two parts:

$$H_{01CV}(k,p) = rkp;$$

$$H_{02CV}(p;c) = -cp + \frac{c^{1-\sigma}}{1-\sigma}; \qquad \text{or} \qquad -cp + \log c$$

where H_{01CV} does not depend on the control c. The Lagrangean L is given by

$$L(k,p;c;q) = H_{0CV}(k,p;c) + kq \qquad k,p,q \in \mathbb{R}, \quad k,q \geq 0;$$

$$c \in [0, +\infty), \quad \text{if } \sigma < 1, \text{ and} \qquad c \in (0, +\infty) \quad \text{if } \sigma \geq 1$$

The maximum value Hamiltonian is, for $k, p \in \mathbb{R}$, $k \geq 0$, if $\sigma < 1$

$$H_0(k,p) = \max_{c \geq 0} H_{0CV}(k,p;c)$$

while, if $\sigma \geq 1$

$$H_0(k,p) = \max_{c > 0} H_{0CV}(k,p;c)$$

(note that we use the notation max instead of sup since we know that the maximum is attained here). If we have

$$p > 0$$

then the maximum point of $H_2(p;c)$ is attained at $c = p^{-1/\sigma}$, so that, for $\sigma \neq 1$,

$$H_0(k,p) = rkp + \frac{\sigma}{1-\sigma} p^{\frac{\sigma-1}{\sigma}}.$$

and, for $\sigma = 1$

$$H_0(k,p) = rkp - 1 - \log p$$

On the contrary, if

$$p = 0$$

then for $\sigma \in (0,1]$

$$H_0(k,p) = +\infty.$$

while, for $\sigma > 1$

$$H_0(k,p) = 0.$$

Finally, if

$$p < 0$$

then, for every $\sigma > 0$,

$$H_0(k,p) = +\infty.$$

For simplicity of notation we define:

$$\begin{aligned} H_{01}(k,p) &= rkp; \\ H_{02}(p) &= \frac{\sigma}{1-\sigma} p^{\frac{\sigma-1}{\sigma}} \text{ or } -1 - \log p, \end{aligned}$$

so that, for $\sigma \in (0,1]$

$$H_0(k,p) = \begin{cases} H_{01}(k,p) + H_{02}(p); & \text{if } p > 0 \\ +\infty; & \text{if } p \leq 0 \end{cases}$$

and, for $\sigma > 1$

$$H_0(k,p) = \begin{cases} H_{01}(k,p) + H_{02}(p); & \text{if } p > 0 \\ 0; & \text{if } p = 0 \\ +\infty; & \text{if } p < 0 \end{cases}.$$

We remark that the maximum point of the Hamiltonian H_{02CV} is always unique (when it exists: but we will prove that this is always the case in section 6.6.3) and it is stricltly positive:

$$\arg\max H_{0CV}(k,p;c) = p^{-1/\sigma} > 0$$

This shows, in particular, that $p > 0$ i.e. that the shadow price of the consumption good is always strictly positive (as it should be by straightforward considerations).

6.6.3 The value function

For a given initial endowement $k_0 > 0$ the value function of the problem (P_σ) is defined as,

$$V(k_0) = \sup_{c \in \mathcal{A}(k_0)} U_\sigma(c).$$

We now devote some space to study properties of V. In fact later we will calculate explicitely V, so this part is not really useful in this case. We keep it to show how to work in case we are not able to calculate V in closed form. We start by a simple result about the class of admissible trajectories

Lemma 6.30 *$\mathcal{A}(k_0)$ is a closed and convex subset of $L^1_{\text{loc}}(0,+\infty;\mathbb{R})$. Moreover, for $\alpha > 0$, $k \in \mathbb{R}_+$*

$$\mathcal{A}(\alpha k) = \alpha \mathcal{A}(k)$$

and, for every $k_1, k_2 \in \mathbb{R}_+$, $\alpha \in (0,1)$

$$k_1 \leq k_2 \Longrightarrow \mathcal{A}(k_1) \subseteq \mathcal{A}(k_2) \tag{41}$$

$$\mathcal{A}(k_1) + \mathcal{A}(k_2) \subseteq \mathcal{A}(k_1 + k_2) \tag{42}$$

$$\mathcal{A}(k_1) \cup \mathcal{A}(k_2) \subseteq \mathcal{A}(k_1 + k_2)$$

$$\alpha \mathcal{A}(k_1) + (1-\alpha)\mathcal{A}(k_2) \subseteq \mathcal{A}(\alpha k_1 + (1-\alpha)k_2) \tag{43}$$

Proof. We omit the proof, since it is immediate from the definitions. ■

Remark 6.31 The converse inclusions hold? Exercise! ■

The following proposition gives useful properties of V.

Proposition 6.32 *Assume that $\rho - r(1-\sigma) > 0$. Then*

(i) For every $k > 0$ we have for $\sigma \in (0,1)$

$$0 \leq V(k) \leq \frac{\rho}{1-\sigma} \frac{\Gamma(1+\sigma)}{a^{1+\sigma}} k^{1-\sigma}$$

while, for $\sigma = 1$

$$-\infty < V(k) \leq \rho \int_t^{+\infty} e^{-\rho s} s \left[rs + \log \frac{k}{s} \right] ds$$

and, for $\sigma \in (1,+\infty)$

$$-\infty < V(k) \leq \frac{\rho}{1-\sigma} \frac{\Gamma(1+\sigma)}{a^{1+\sigma}} k^{1-\sigma}$$

(ii) V is increasing and subadditive in the sense that

$$k_1 \leq k_2 \Longrightarrow V(k_1) \leq V(k_2) \quad \forall k_1, k_2 \in \mathbb{R}_+$$

$$V(k_1 + k_2) \leq V(k_1) + V(k_2) \quad \forall k_1, k_2 \in \mathbb{R}_+.$$

(iii) For $\sigma \neq 1$ V is $(1-\sigma)$-homogeneous in the sense that

$$V(\alpha k) = \alpha^{1-\sigma} V(k) \quad \forall \alpha > 0,\ k \in \mathbb{R}_+$$

and for $\sigma = 1$

$$V(\alpha k) = \rho^{-1} \log \alpha + V(k) \quad \forall \alpha > 0,\ k \in \mathbb{R}_+$$

(iv) V is continuous on the positive semiaxis $\mathbb{R}_+$ (included the boundary if $\sigma \in (0,1)$) and concave.

(v) V is two times differentiable a.e. on $\mathbb{R}_+$ and $V' > 0$ at every point of differentiability. Moreover V admits non-empty superdifferential at every point of $\mathbb{R}_+$ and $D^+V(k) \subseteq \mathbb{R}_+$ for every $k \in \mathbb{R}^n_+$.

Proof.
Proof of (i). It easily follows by the previous proposition.
Proof of (ii). Since $\mathcal{A}(\alpha k) = \alpha \mathcal{A}(k)$ we have

$$V(\alpha k) = \sup_{c \in \mathcal{A}(\alpha k)} \int_0^{+\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt = \sup_{c \in \mathcal{A}(k)} \int_0^{+\infty} e^{-\rho t} \frac{[\alpha c(t)]^{1-\sigma}}{1-\sigma} dt = \alpha^{1-\sigma} V(k)$$

The monotonicity follows directly from (41) and we omit it.
Proof of (iii). For $\alpha \in (0,1)$ we have

$$\begin{aligned}
& \alpha V(k_1) + (1-\alpha) V(k_2) \\
= \ & \alpha \sup_{c \in \mathcal{A}(k_1)} \int_0^{+\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt + (1-\alpha) \sup_{c \in \mathcal{A}(k_2)} \int_0^{+\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \\
= \ & \sup_{\substack{c_1 \in \mathcal{A}(k_1), \\ c_2 \in \mathcal{A}(k_2)}} \int_0^{+\infty} e^{-\rho t} \frac{\alpha c_1(t)^{1-\sigma} + (1-\alpha) c_2(t)^{1-\sigma}}{1-\sigma} dt \\
\leq \ & \sup_{\substack{c_1 \in \mathcal{A}(k_1), \\ c_2 \in \mathcal{A}(k_2)}} \int_0^{+\infty} e^{-\rho t} \frac{[\alpha c_1(t) + (1-\alpha) c_2(t)]^{1-\sigma}}{1-\sigma} dt
\end{aligned}$$

which implies, by (43) that

$$\begin{aligned}
\alpha V(k_1) + (1-\alpha) V(k_2) &\leq \sup_{c \in \mathcal{A}(\alpha k_1 + (1-\alpha) k_2)} \int_0^{+\infty} e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \\
&= V(\alpha k_1 + (1-\alpha) k_2)
\end{aligned}$$

which gives the concavity. The continuity up to the boundary follows by applying standard results on convex functions (see e.g. [16], [35]).

Proof of (iv). This part follows in the interior by applying standard results on convex functions (see e.g. [16], [35]) while on the boundary one needs to use the definition of V and argue as in the proof of Theorem **??**; we omit this part for brevity. ■

The following Proposition is a version of the Bellman Optimality Principle (BOP) (see e.g. [8], [20], [4]) for our example. We have shown it only for an example of BOP in the case of infinite horizon with discount.

Proposition 6.33 *Assume that Hypothesis* **??** *hold. For every $t \geq 0$ we set $\mathcal{A}_t(k_0)$ as the set of control strategies that satisfies all the constraints on c and k up to time t, and*

$$J_t(c) = \int_0^t e^{-\rho s} \frac{c(s)^{1-\sigma}}{1-\sigma} ds + e^{-\rho t} V(k(t; 0, k_0, c))$$

Then, for every $c \in \mathcal{A}(k_0)$ the function $t \to g(t) = J_t(c)$ is nonincreasing and we have, for every $t \geq 0$

$$V(k_0) = \sup_{c \in \mathcal{A}_t(k_0)} J_t(c) \tag{44}$$

Moreover, if c is optimal for (P_σ) then its restriction to $[0,t]$ is optimal for the problem $(P_{t,\sigma})$ of maximizing $J_t(c)$ and the function $t \to g(t)$ is constant.

Proof. The proof is a standard (see e.g. [8], [20], [4]). ■

The Hamilton-Jacobi equation associated to our problem is

$$\rho u(k) - H_0(k, u'(k)) \qquad \forall k \geq 0. \tag{45}$$

where we recall that, in the case $\sigma \in (0,1)$

$$\begin{aligned} H_0(k,p) &= \sup_{c \geq 0} \left\{ rkp - cp + \frac{c^{1-\sigma}}{1-\sigma} \right\} \\ &= rkp + \frac{\sigma}{1-\sigma} p^{\frac{\sigma-1}{\sigma}}. \end{aligned}$$

The Dynamic Programming Principle (44) imply that the value function V is the unique solution of the above equation in the sense of viscosity solutions (see e.g. [17], [5]). Moreover, by using arguments of [4] we can prove that V is also C^1 and so it is a classical solution of equation (45). However in this case we will get explicit solution so do not go deep in this regularity problems.

Proposition 6.34 *Assume that Hypothesis* **??** *hold. Then the value function V is a classical (and also viscosity) solution of the equation (45).*

We observe that (see e.g. [4, p. 133]) from the dynamic programming principle (Proposition 6.33) the following optimality condition follows: a control strategy $\hat{c} \in \mathcal{A}(k_0)$ is optimal for the problem (P_σ) if and only if the function

$$g(t) = \int_0^t e^{-\rho s} \frac{\hat{c}(s)^{1-\sigma}}{1-\sigma} ds + e^{-\rho t} V(k(t; 0, k_0, \hat{c}))$$

is nondecreasing for $t \geq 0$. This fact, together with Proposition 6.34 above implies the following necessary condition of optimality, which is in fact a form of the so-called Pontryagin Maximum Principle.

Theorem 6.35 *Assume that Hypothesis* **??** *hold. Assume also that $\hat{c} \in \mathcal{A}(k_0)$ is optimal for the problem (P_σ) and let $\hat{k}$ be the corresponding optimal state. Then, for a.e. $t \geq 0$,*

$$\frac{\hat{c}(t)^{1-\sigma}}{1-\sigma} - \rho V\left(\hat{k}(t)\right) + \hat{k}'(t) V'\left(\hat{k}(t)\right) \hat{k}'(t) = 0$$

i.e.

$$r\hat{k}(t) V'\left(\hat{k}(t)\right) - \hat{c}(t) V'\left(\hat{k}(t)\right) + \frac{\hat{c}(t)^{1-\sigma}}{1-\sigma} = \rho V\left(\hat{k}(t)\right)$$

and also

$$\begin{aligned} & -\hat{c}_t V'\left(\hat{k}(t)\right) + \frac{\hat{c}(t)^{1-\sigma}}{1-\sigma} \\ = & \sup_{c \geq 0} \left\{ -cV'\left(\hat{k}(t)\right) + \frac{c^{1-\sigma}}{1-\sigma} \right\}. \end{aligned}$$

In particular V admits first derivative and

$$\hat{c}(t)^{-\sigma} = V'\left(\hat{k}(t)\right)$$

Proof. It is enough to apply the same argument of [5] and [39] (see also [4, p.133-136]) adapted to this case. We omit it for brevity. ■

The above Theorem 6.35 gives the feedback formula we need. Now we show how to calculate the optimal control in this case.

First of all we observe that the function $v(k) = ak^{1-\sigma}$, with $a = \frac{1}{1-\sigma}\left[\frac{\rho - r(1-\sigma)}{\sigma}\right]^{-\sigma}$, is a classical solution of the HJB equation (45), it is enough to make a substitution. Then, using Theorem 6.24 we get that $v \geq V$. Moreover, if we consider the feedback control

$$c(s) = v'(k(s))^{-1/\sigma} = [a(1-\sigma)]^{-1/\sigma} k(s) = \frac{\rho - r(1-\sigma)}{\sigma} k(s)$$

then the closed loop equation

$$\begin{aligned} k'(s) &= rk(s) - v'(k(s))^{-1/\sigma} = -\frac{\rho - r}{\sigma} k(s); \\ k(0) &= k_0 \end{aligned}$$

is linear and has a unique solution

$$k(s) = e^{s(r-\rho)/\sigma} k_0 \geq 0$$

so that

$$c(s) = [a(1-\sigma)]^{-1/\sigma} e^{s(r-\rho)/\sigma} k_0 \geq 0.$$

Since the couple (k, c) satisfy the admissibility constraints, then it is optimal. For economic comments see [6].

6.7 DP method in the optimal investment problem

Let us consider the classical optimal investment problem with quadratic adjustment costs and a linear technology:

$$\max J(k_0; u) = \max \int_0^{+\infty} e^{-\rho t}[ak(t) - bu(s) - \frac{c}{2}u^2(t)]dt,$$

$$\dot{k}(t) = u(t) - \mu k(t), \quad k(0) = k_0,$$

$a > 0,\ b > 0,\ c > 0$, subject to the usual constraint $k \geq 0$. u denotes investments and k is the stock of capital.

Set $\overline{a} = a/(\rho + \mu)$, the expected return from a unit of capital. Assume that $\overline{a} \geq b$ (which means that investments are profitable) and choose measurable control strategies u such that $t \mapsto e^{-\rho t}u^2(t)$ are square integrable and the state constraint $k \geq 0$ is satisfied: this means that

$$\mathcal{U}_{ad}(k_0) = \left\{u \in L^2_\rho(0, +\infty) : \quad k(s; k_0, u) \geq 0, \quad s \geq 0\right\}.$$

The value function is

$$V(k_0) = \sup_{u \in \mathcal{U}_{ad}(k_0)} J(k_0; u)$$

The current value Hamiltonian is defined as

$$\begin{aligned} H_{0CV}(k, p, u) &= (-\mu k + u)p + ak - bu - \frac{c}{2}u^2 = [-\mu kp + ak] + \left[up - bu - \frac{c}{2}u^2\right] \\ &\stackrel{def}{=} H_{01CV}(k, p) + H_{02CV}(p; u) \end{aligned}$$

and the maximum value Hamiltonian as

$$H_0(k, p) = \sup_{u \in \mathbb{R}} H_{0CV}(k, p; u) = [-\mu kp + ak] + \left[\frac{(p-b)^2}{2c}\right] \stackrel{def}{=} H_{01}(k, p) + H_{02}(p),$$

where the maximum point is reached at $u = (p - b)/c$. The HJB equation is

$$\rho v(k) = -\mu k Dv(k) + ak + H_{02}(Dv(k)); \qquad k \geq 0. \tag{46}$$

We observe that a regular solution of it is the linear function

$$v(k) = \overline{a}k + \frac{1}{\rho} H_{02}(\overline{a}).$$

Similarly to the previous example it can be proved that the value function V is the unique viscosity solution (here also classical) of the HJB equation (46) so $v = V$. The optimal control can the n be easily found by solving the closed loop equation. We give the explicit form of the optimal couple in the exercise below suggesting an alternative method to compute it.

EXERCISE: substitute $k(s; k_0, u)$ inside the functional and get directly that the optimal control-state trajectory for the problem is

$$u^*(t) \equiv \frac{1}{c}\left[\overline{a} - b\right], \quad k^*(t) = \frac{u^*}{\mu} + e^{-\mu t}\left[k_0 - \frac{u^*}{\mu}\right], \tag{47}$$

and the value function is

$$V(k_0) = \frac{ak_0}{\rho + \mu} + \frac{1}{2c\rho}\left[\overline{a} - b\right]^2.$$

7 Optimality Conditions: the Maximum principle

The problem of finding "good" (i.e. that can be handled in a big variety of cases) optimality conditions is the key point of all optimization problems. In fact also Dynamic Programming gives also optimality conditions (see e.g. Theorem 6.24), but we are interested here to more classical optimality conditions that are in a sense the infinite dimensional analogous of the Kuhm-Tucker conditions (obtained by the Lagrange multiplier method): the Pontryagin Maximum Principle (PMP). This is the most popular optimality condition and in fact the name PMP is now more or less a synonymous of Necessary Optimality Conditions for Optimal Control Problems. As for the Dynamic Programming we will not give a complete treatment of it: we will just recall the PMP in a special case and show how to apply it to our examples.

We start by the discrete time case and then we pass to the continuous time case.

7.1 PMP: discrete time case

We derive a version of the Pontryagin Maximum Principle for a very simple control problem, where controls and trajectories are real valued (respectively, $c(t) \in \mathbb{R}$, $x(t) \in \mathbb{R}$) and no constraints are assumed. We indeed maximize the functional

$$J(t_0, x_0; c(\cdot)) = \sum_{t=0}^{T-1} f_0(t, x(t), c(t)) + \phi(x(T))$$

under the constraints

$$x(t+1) = f(t, x(t), c(t)) \quad \forall t = 0, 1, .., T-1, \tag{48}$$

subject to the initial condition

$$x(t_0) = x_0.$$

For simplicity we set $t_0 = 0$ so $\mathcal{T} = \{0, 1, \ldots, T-1\}$. Rather than thinking of controls and trajectories as sequences, we consider them as vectors in $\mathbb{R}^T$

$$\mathbf{c} = (c(0), c(1), .., c(T-1)), \quad \mathbf{x} = (x(1), .., x(T-1), x(T)),$$

Under this perspective, the problem is a static optimization problem in $2T$ variables, with T equality constraints given by (48). The problem can be solved by means of Lagrange Theorem, provided the assumptions of the theorem hold, where the Lagrangian is built by introducing the T multipliers

$$\mathbf{p} = (p(1), .., p(T-1), p(T))$$

as follows

$$L(\mathbf{x}, \mathbf{c}, \mathbf{p}) = \sum_{t=0}^{T-1} f_0\left(t, x(t), c(t)\right) + \phi(x(T)) - \sum_{t=0}^{T-1} p(t+1)\left(x(t+1) - f_D\left(t, x(t), c(t)\right)\right)$$

If $(\mathbf{x}^*, \mathbf{c}^*)$ is the solution to the problem, then necessarily there exists a vector $\mathbf{p}^* \in \mathbb{R}^{\mathbf{T}}$ such that for all $t = 0, 1, \ldots, T-1$ one has

$$\begin{cases} L'_{x(t)}(\mathbf{x}^*, \mathbf{c}^*, \mathbf{p}^*) = \mathbf{0} \\ L'_{c(t)}(\mathbf{x}^*, \mathbf{c}^*, \mathbf{p}^*) = \mathbf{0} \\ L'_{p(t)}(\mathbf{x}^*, \mathbf{c}^*, \mathbf{p}^*) = \mathbf{0} \end{cases}$$

The Maximum principle is nothing but Lagrange theorem rephrased into term of the Hamiltonian function.

The Hamiltonian function. We define as **current value Hamiltonian** the function $H_{CV} : \mathcal{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$H_{CV}(t, x, p; c) \doteq f_0(t, x, c) + p f_D(t, x, c), \qquad t = 0, .., T-1$$

(note that here x, c, p are variables in $\mathbb{R}$). Note that

$$\begin{aligned} L(\mathbf{x}, \mathbf{c}, \mathbf{p}) &= \sum_{t=0}^{T-1} f_0\left(t, x(t), c(t)\right) + \phi(x(T)) - \sum_{t=0}^{T-1} p(t+1)\left(x(t+1) - f_D\left(t, x(t), c(t)\right)\right) \\ &= \sum_{t=0}^{T-1} H_{CV}\left(t, x(t), c(t), p(t+1)\right) - \sum_{t=1}^{T-1} p(t+1)x(t+1) + \phi(x(T)) \end{aligned}$$

Hence the necessary conditions may be written as

$$\begin{cases} p(t) = H'_x(t, x(t), c(t), p(t+1)) & t = 1, .., T-1 \\ p(T) = \phi'_x(x(T)) & t = T \\ H'_c(t, x(t), c(t), p(t+1)) = 0 & t = 0, .., T-1 \\ x(t+1) = H'_p(t, x(t), c(t), p(t+1)) & t = 0, .., T-1 \\ x(0) = x^0 \end{cases} \tag{49}$$

where we added the initial condition on the trajectory (note that the last $T+1$ equations are the state equation(s)).

Theorem 7.1 *(Pontryagin Maximum Principle) Assume* $(x^*(t), c^*(t))$ *is an optimal couple for the assigned problem and let* H *be defined as above. Then there exists a vector* $\mathbf{p}^* = (p^*(1), .., p^*(T))$*, such that the system (49) is satisfied.*

Remark 7.2

1. *The function* $p(t)$ *is often called* adjoint variable *or* costate*.*
2. *The condition* $H'_c(t, x(t), c(t), p(t)) = 0$ *is indeed generalized by*

$$c(t) \in \arg\max \{H(t, x(t), c, p(t+1)) : c \in \mathbb{R}\}, \text{ for all } t \in \mathcal{T}$$

3. *If the objective functional is of type*

$$J(\mathbf{c}(\cdot)) = \sum_{t=0}^{T} l(t, x(t), c(t))$$

   then the problem has horizon $T+1$ *and zero final objective* $\varphi \equiv 0$*, so that the final condition on the costate is* $p(T+1) = 0$*.*

4. *In the infinite horizon case still the above conditions are necessary but without the terminal condition on the costate* p*.*
5. *The theorem gives necessary conditions of optimality. Nevertheless (49) turns to be sufficient under some concavity properties of the Hamiltonian, as we see next.*

Theorem 7.3 *(Sufficient condition of optimality). Assume that the triple* $(x(t), c(t), p(t))$ *satisfies the conditions (49) of the Maximum Principle, and assume* $H(t, x, c, p(t))$ *is concave with respect to* (x, c) *for every* t*. Then the triple* $(x(t), c(t), p(t))$ *is optimal.*

Exercises

1. Maximize

$$\sum_{t=0}^{2}(1+x(t)-c^2(t))+\varphi(x(3))$$

under the condition

$$\begin{cases} x\,(t+1)=x\,(t)+c(t) & t=\prime,\infty,\in \\ x\,(0)=0, \end{cases}$$

when:

(a) $\varphi(x)=x$

(b) $\varphi(x)=-x^2$

Solution. The horizon is $T=3$; the Hamiltonian $H(t,x,c,p)=1+x-c^2+p(x+c)$,

$t=\prime,\infty,\in$. Note that H is concave in (c,x) so that (PMP) gives necessary and sufficient conditions.

(a) Since

$$H'_x=1+p;\quad H'_c=-2c+p;\quad H'_p=x+c;\quad \varphi'=-x^2$$

the Hamiltonian system is

$$\begin{cases} p(t)=1+p(t+1) & t=\infty,\in \\ p(3)=1 & t=3 \\ -2c(t)+p(t+1)=0 & t=\prime,\infty,\in \\ x\,(t+1)=x\,(t)+c(t) & t=\prime,\infty,\in \\ x\,(0)=0 & t=0 \end{cases}$$

which soluton is

$$\begin{aligned} p^*(3) &= 1,\quad p^*(2)=1+p^*(3)=2,\quad p^*(1)=1+p^*(2)=3 \\ c^*(0) &= \frac{p^*(1)}{2}=\frac{3}{2},\quad c^*(1)=\frac{p^*(2)}{2}=1,\quad c^*(2)=\frac{p^*(3)}{2}=\frac{1}{2} \\ x^*(1) &= x(0)+c^*(0)=\frac{3}{2},\quad x^*(2)=x^*(1)+c^*(1)=\frac{5}{2},\quad x^*(3)=x^*(2)+c^*(2)=3 \end{aligned}$$

(b) Here the final objective is $\varphi=-x^2$, so that $\varphi'=-2x$ and the final condition on the costate is $p(4)=-2x(3)$.The PMP gives

$$\begin{aligned} p(3) &= -2x(3),\quad p(2)=1-2x(3),\quad p(1)=2-2x(3) \\ c(0) &= 1-x(3),\quad c(1)=\frac{1}{2}-x(3),\quad c(2)=-x(3) \\ x(1) &= 1-x(3),\quad x(2)=\frac{3}{2}-2x(3),\quad x(3)=\frac{3}{2}-3x(3) \end{aligned}$$

so that

$$\begin{aligned} x^*(1) &= \frac{5}{8}, \quad x^*(2) = \frac{3}{4}, \quad x^*(3) = \frac{3}{8} \\ p^*(3) &= -\frac{3}{4}, \quad p^*(2) = -\frac{1}{4}, \quad p^*(1) = \frac{5}{4} \\ c^*(0) &= \frac{5}{8}, \quad c^*(1) = \frac{1}{8}, \quad c^*(2) = -\frac{3}{8} \end{aligned}$$

2. Maximize

$$\sum_{t=0}^{3}(1 + x(t) - c^2(t))$$

under the condition

$$\begin{cases} x\,(t+1) = x\,(t) + c(t) & t=\prime, \infty, \in \\ x\,(0) = 0, \end{cases}$$

Solution. With reference to the previous exercise, now the horizon is $T = 4$, the final objective $\varphi \equiv 0$, so that the final condition on the costate is $p(4) = 0$. The solution is

$$\begin{aligned} p^*(4) &= 0, \quad p^*(3) = 1, \quad p^*(2) = 2, \quad p^*(1) = 3 \\ c^*(0) &= \frac{3}{2}, \quad c^*(1) = 1, \quad c^*(2) = \frac{1}{2}, \quad c^*(3) = 0 \\ x^*(1) &= \frac{3}{2}, \quad x^*(2) = \frac{5}{2}, \quad x^*(3) = 3, \quad x^*(4) = 3 \end{aligned}$$

3. Maximize

$$\sum_{t=0}^{2}(1 + -x^2(t) - 2c^2(t))$$

under the condition

$$\begin{cases} x\,(t+1) = x\,(t) - c(t) & t=\prime, \infty, \in \\ x\,(0) = 5, \end{cases}$$

Solution. PMP gives necessary and sufficient conditions of optimality since the Hamiltonian $H = -x^2 - 2c^2 + p(x - c)$ is concave in (x, c)

$$\begin{cases} p(t) = -2x(t) + p(t+1) & t=\infty, \in \\ p(3) = 0 & t = 3 \\ -4c(t) - p(t+1) = 0 & t=\prime, \infty, \in \\ x\,(t+1) = x\,(t) - c(t) & t=\prime, \infty, \in \\ x\,(0) = 5 & t = 0 \end{cases}$$

that implies

$$\begin{aligned} p^*(3) &= 0, \quad p^*(2) = , \quad p^*(1) = \\ c^*(0) &= \frac{25}{11}, \quad c^*(1) = \frac{10}{11}, \quad c^*(2) = 0, \\ x^*(1) &= \frac{30}{11}, \quad x^*(2) = \frac{20}{11}, \quad x^*(3) = \frac{20}{11} \end{aligned}$$

7.2 PMP: continuous time case

We now pass to the continuous time case. We need more assumptions listed here.

Hypothesis 7.4 *The functions f_0 and f_C satisfy the assumptions of Theorem 6.19 and moreover are Fréchet differentiable in x.*

Hypothesis 7.5 *There are no state constraints, so $X = \mathbb{R}^n$.*

Theorem 7.6 (PMP) *Assume that Hypotheses 3.12, 7.4, 7.5 hold for problem (P) and let $T < +\infty$. Fix $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ and let $(x(\cdot), u(cdot))$ be an optimal state-control couple for problem (P) starting at (t_0, x_0). Then there exists an absolute continuous function $p : [t,T] \mapsto \mathbb{R}^n$ such that:*

1. *p is a solution of the* backward *O.D.E.*

$$\begin{aligned} p'(s) &= -\left[f_{C,x}(s, x(s), c(s))\right]^* p(s) - f_{0,x}(s, x(s), c(s)); \qquad s \in [t,T] \quad (50) \\ p(T) &= D\phi(x(T)); \end{aligned}$$

2. *for every $s \in [t,T]$*

$$c(s) \in \arg\max_{c \in C} \{H_{CV}(s, x(s), p(s); c)\} \quad (51)$$

Proof. see e.g [42]. ■

Remark 7.7 *Observe that equation (51) is the precise statement of Pontryagin Maximum Principle and it is perfectly analogous to the condition (ii) of Theorem 6.24 with $p(s)$ in place of $D_x V(s, x(s))$. Here $p(\cdot)$ is obtained as a solution (better if unique) of the* backward *O.D.E. (50). Note that the final condition $p(T) = D\phi(x(T))$ is coherent with the interpretation of $p(s)$ as $D_x V(s, x(s))$. Finally we recall that the final condition $p(T) = D\phi(x(T))$ is called trasversality condition, the reason of this name is due to the fact that, in the variational formulation, it states the orthogonality of the optimal trajectory to the final target set (see e.g. [19], [23]).*

Remark 7.8 *The auxiliary variable $p(\cdot)$ plays the role of a Lagrange multiplier in infinite dimension here. It will be called the co-state variable (or the dual variable) of the problem (P) and it can be interpreted in many economic problems (like the ones in our examples) as the shadow price of the capital good. Due to its important meaning we will generally be interested also in finding p. So we can talk about an optimal triple $(x(\cdot), p(\cdot), c(\cdot))$ when $(x(\cdot), c(\cdot))$ is an optimal couple and $p(\cdot)$ is the corresponding co-state.*

Remark 7.9 *The idea of the proof comes from the multiplier method used in static optimization. In fact the above Theorem is just a special case of the wide quantity of necessary conditions for dynamic optimization problems, see on this [31], [37], [27].*

The above conditions become sufficient conditions under some more assumptions, e.g. concavity of the problem, see on this [37]. Here we limit ourselves to the following result

Theorem 7.10 *Assume that Hypotheses 3.12, 7.4, 7.5 hold for problem (P) and let $T < +\infty$. Fix $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ and let $(x(\cdot), c(\cdot))$ be an admissible state control couple for problem (P) starting at (t_0, x_0). Assume that f_0 and f_C are concave in (x, c) and that there exists an absolute continuous function $p : [t,T] \mapsto \mathbb{R}^n$ such that:*

1. *$p(\cdot)$ is a solution of the* backward *O.D.E.*

$$\begin{aligned} p'(s) &= -\left[f_{C,x}(s, x(s), c(s))\right]^* p(s) - f_{0,x}(s, x(s), c(s)); \qquad s \in [t,T] \quad (52) \\ p(T) &= D\phi(x(T)); \end{aligned}$$

2. *for every $s \in [t,T]$*

$$c(s) \in \arg\max_{c \in C} \{H_{CV}(s, x(s), p(s); c)\} \quad (53)$$

Then $(x(\cdot), c(\cdot))$ is optimal.

Proof. see e.g [42]. ■

Remark 7.11 *The PMP substantially says that, to find optimal couples we need to solve the system*

$$\begin{aligned} x'(s) &= f(s, x(s), c(s)); \qquad x(t_0) = x_0 \\ p'(s) &= -\left[f_{C,x}(s, x(s), c(s))\right]^* p(s) - f_{0,x}(s, x(s), c(s)); \qquad p(T) = D\phi(x(T)) \\ c(s) &\in \arg\max_{c \in C} \{H_{CV}(s, x(s), p(s); c)\}. \end{aligned}$$

that can be rewritten as (note that H_{CV} is always differentiable in p, and, thanks to Hypothesis 7.4 it is also differentiable in x)

$$\begin{aligned} x'(s) &= \frac{\partial}{\partial p} H_{CV}(s, x(s), p(s); c(s)); \qquad x(t_0) = x_0 \quad (54) \\ p'(s) &= -\frac{\partial}{\partial x} H_{CV}(s, x(s), p(s); c(s)); \qquad p(T) = D\phi(x(T)) \\ c(s) &\in \arg\max_{c \in C} \{H_{CV}(s, x(s), p(s); c)\}. \end{aligned}$$

If we know that, for every (t, x, p) there exists an interior maximum point of $H_{CV}(t, x, p; c)$, and that $H_{CV}(t, x, p; c)$ is differentiable with respect to c, then the third condition can be substituted by the weaker one

$$\frac{\partial}{\partial c} H_{CV}(s, x(s), p(s); c(s)) = 0.$$

Moreover, since $H_{CV}(s, x(s), p(s); c(s)) = H_{MAX}(s, x(s), p(s))$, it can be proved that, if also H_{MAX} is differentiable the system (54) is equivalent to

$$
\begin{aligned}
x'(s) &= \frac{\partial}{\partial p} H_{MAX}(s, y(s), p(s); c(s)); \qquad x(t_0) = x_0 \qquad (55)\\
p'(s) &= -\frac{\partial}{\partial x} H_{MAX}(s, x(s), p(s); c(s)); \qquad p(T) = D\phi(x(T))\\
c(s) &\in \arg\max_{c \in C} \{H_{CV}(s, x(s), p(s); c)\}.
\end{aligned}
$$

Such system will be called Hamiltonian system of our control problem (P).

Solving such system will give candidates (y, p, u) for being an optimal triple. Note that it is hard to solve such systems, due to the trasversality condition $p(T) = D\phi(x(T))$. If we have an initial condition on $p(\cdot)$ then we could use standard existence and uniqueness theorems for systems of ODE. Here the trasversality condition is backward and moreover it depends on the final value of the other variable of the system.

Remark 7.12 *Autonomous infinite horizon case with discount. In this case $f_0(t, x, c) = e^{-\rho t} f_1(x, c)$ and $f_C(t, x, c)$ does not depend on t. The co-state equation becomes*

$$
\begin{aligned}
p'(s) &= [\rho - [f_{C,x}(x(s), c(s))]^*] p(s) - f_{1,x}(x(s), c(s)); \qquad s \in [t, T] \qquad (56)\\
p(+\infty) &= ???????
\end{aligned}
$$

The problem is the condition at infinity. There is no good condition. Under some additional assumptions there are some necessary conditions at infinity but there is no universal way to set up them. We mention two of them:

$$
\lim_{t \to +\infty} e^{-\rho t} p(t) = 0, \qquad \lim_{t \to +\infty} e^{-\rho t} x(t) \cdot p(t) = 0
$$

but also other are possible. See e.g. on this [2, 10, 28].

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