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ON PHASE-TYPE DISTRIBUTIONS IN RUIN THEORY

Pietroluongo Mariafortuna[1]

Abstract

The aim of this paper is to serve as an introduction to the use of phase-type distributions and at the same time to outline their use in ruin theory. Phase-type distributions, a particular class of matrix-exponential distributions, have the important advantage of being suitable for approximating most of other distributions as well as being mathematically tractable.
After a review on phase-type distributions and their properties, a possible use in ruin theory is illustrated. Modelling both interarrival claim times and individual claim sizes with this class of distributions an explicit formula for the probability of ultimate ruin is given.

Keywords

Phase-type distribution, Markov chain, Ruin theory, Ruin probability.

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1. Introduction

A phase-type distribution is defined as the distribution of absorption times of certain Markov jump processes. They constitute a class of distributions which seems to strike a balance between generality and tractability. Indeed, any positive distribution may be approximated arbitrarily closely by phase-type distributions whereas exact solutions to many complex problems in stochastic modeling can be obtained either explicitly or numerically.

The use of phase-type distributions has led to a wide range of stochastic modeling applications which are algorithmically tractable in areas as diverse as queueing theory, ruin theory, telecommunications, biostatistics, drug kinetics, and survival analysis.

The family of phase-type distributions has gained widespread attention in the area of stochastic modelling, particularly when Markov processes are involved, since they are one of the most general classes of distributions permitting a Markovian interpretation.

There are several reasons to use phase-type distributions. First of all, because they are quite flexible in terms of their possible shapes and because their inherent mathematical and numerical tractability.

Moreover, phase-type distributions are dense in the set of all distributions, so that - in principle - one can replace any (non-phase-type) distribution with a suitable phase-type approximation. It should be taken into account that, since they have exponentially decreasing tails, they can not be used for large or extreme value problems.

One of the most useful features of this class of distributions is that they allow for the use of matrix-analytic methods in stochastic models. Using these methods, numerical integrations arising in the study of many stochastic models are replaced by matrix operations that develop naturally in the analysis of structured Markov chains, being matrix exponentials nowadays easy to calculate. Many results using phase-type methodologies have been generalized into the broader class of matrix-exponential distributions, with a rational Laplace transform.

A short bibliographic review could begin with Erlang (1909), but the major contribution is due to Neuts (1981, 1995). Phase-type distributions are used in many different fields of applications, so there is a large number of papers about this topic. Concerning risk theory, Asmussen (2000, 2003), Asmussen and Bladt (1996) and Bladt (2005) have given many results using phase-type methodologies. More recently, among others, we can mention Hipp (2006), Ahn and Badescu (2007) and Jang (2007).

In the next Section 2 we recall the fundamentals of continuous time Markov processes with finite state spaces. In section 3 we we introduce the Phase – Type distribution and the notation

we use further in the paper, giving some examples in section 4. The last section 5 contains some applications in risk theory.

2. Markov jump processes

Consider a continuous time stochastic process $\{X(t)\}_{t\in\Re_0^+}$ taking on values in a set of non-negative integers (state space).

The process $\{X(t)\}_{t\in\Re_0^+}$ is a continuous time Markov chain if for all s,t≥0 and nonnegative integers i, j, x(u), 0≤u<s,

$$\text{Prob}\,\{X(t+s)=j \mid X(s)=i, X(u)=x(u), 0\le u<s\} = \text{Prob}\,\{X(t+s)=j \mid X(s)=i\}\ .$$

In other words, a continuous time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future state at time t+s, given the present state at s and all past states, depends only on the present state and is independent of the past.

For our purposes, we consider a finite state space E={1,2,…,n}. Let T_1, T_2, … denote the times where $\{X(t)\}_{t\in\Re_0^+}$ jumps from one state to another, being $T_0=0$.

Then the discrete time process $\{Y(n)\}_{n\in N}$, where $Y(n)=X(T_n)$ is a Markov chain describing the visited states with transition matrix $\mathbf{Q}=\{q_{ij}\}_{i,\,j\in E}$, where q_{ij} is the probability that process goes from state i to state j.

If Y(n)=i, $\tau_i=T_{n+1}-T_n$ is the amount of time that the process stays in state i before making a transition into a different state; then for all s,t ≥0

$$\text{Prob}\{\tau_i>s+t \mid \tau_i>s\} = \text{Prob}\{\tau_i>t\}\ .$$

Hence, the random variable τ_i is memoryless and must be exponentially distributed with a certain parameter λ_i.

So, a continuous time Markov chain is a stochastic process that moves from state to state in accordance with a (discrete time) Markov chain, but it is such that the amount of time it spends in each state, before proceeding to the next state, is exponentially distributed. In addition the amount of time the process spend in state i, and the next state visited, must be independent random variables.

Since $\lambda_i dt$ is the probability that the process leaves state i during the infinitesimal time interval [t,t+dt), it follows that

$$\lambda_{ij} = \lambda_i\, q_{ij} \qquad (i\neq j)$$

is the intensity of jumping from state i to state j.

Define the intensity matrix or infinitesimal generator of the process as

$$\mathbf{\Lambda} = \{\lambda_{ij}\}_{i,\,j\in E}\ ,$$

where

$$\lambda_{ij} = \lambda_i\, q_{ij} \qquad (i\neq j)\ ,$$
$$\lambda_{ii} = -\sum_{h\neq i}\lambda_{ih} \qquad (i=j)\ .$$

Denote by q^t_{ij} the probability that a Markov chain, presently in state i, will be in state j after an additional time t, that is

$$q^t_{ij} = \text{Prob}\,\{X(t+s)=j \mid X(s)=i\}\ ,$$

and by $\mathbf{Q}^t = \{q^t_{i,j}\}_{i,j\in E}$ the corresponding transition matrix.

Then

$$\mathbf{Q}^t = \exp(\mathbf{\Lambda} t)\ ,$$

where the exponential of a p×p matrix $\mathbf{\Lambda}$ is defined by the series expansion $\exp(\Lambda) = \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!}$.

Let f_{ii} be the probability that, starting in state i, the process will ever reenter state i.

Defining with $f^n_{ii} = \text{Prob}\{X(n)=i;\ X(k)\neq i,\ k=1,\dots,n-1 \mid X(0)=i\}$ (with $f_{ii}^0=0$) the probability that starting in i the first transition into i occurs at time n, it follows that

$$f_{ii} = \sum_{n=1}^{\infty} f^n_{ii}\ .$$

If $f_{ii}=1$, the state is *recurrent*, otherwise it is *transient*.

If state i is recurrent then, starting from state i, the process will reenter state i infinitely often with probability 1.

If state i is transient then, starting in state i, the number of periods in which the process is in state i has a geometric distribution with finite mean $1/(1-f_{ii})$.

Equivalently, state i is recurrent if $\sum_{n=1}^{\infty} q^n_{ii} = \infty$ and transient if $\sum_{n=1}^{\infty} q^n_{ii} < \infty$.

So, a transient state will only be visited a finite number of times (hence the name transient) and in a finite state Markov chain not all states can be transient.

A special case of a recurrent state is if $q_{ij}=0$ for all i≠j, implying $\lambda_{ij}=0$ for all j, (or $q_{ii}=1$) then i is *absorbing*.

3. Phase-type distributions

A phase-type distribution of order p is defined as the absorption time distribution in a finite state Markov process with p transient states and one absorbing state.

Let $\{X(t)\}_{t\in\Re_0^+}$ be a Markov jump process on a finite state space $\tilde{E}=E\cup\{p+1\}$, $E=\{1,2,\dots,p\}$, where states 1, …, p are transient and state p+1 is absorbing. This implies that the intensity matrix of $\{X(t)\}_{t\in\Re_0^+}$ can be written in block partitioned form as:

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0}' & 0 \end{pmatrix}$$

where $\mathbf{T}$ is a p×p dimensional matrix, $\mathbf{t}$ and $\mathbf{0}$ are two vectors, with dimensions p×1.
Since the intensity matrix of a non terminating Markov process has rows that sum to zero, it is:

$$\mathbf{t}+\mathbf{Te}=0 \Leftrightarrow \mathbf{t}=-\mathbf{Te} \qquad \text{where } \mathbf{e}=(1, 1, \ldots, 1)'$$

The interpretation of vector $\mathbf{t}$ is as the exit rate (exit from the transient subset of states E) vector, since the intensities t_i are the intensities by which the process jumps to the absorbing state.
Now we define the initial probabilities as $\pi_i=\text{Prob}\{X_0=i\}$, $i\in E$, and $\text{Prob}\{X_0=p+1\}=0$ meaning that the process cannot initiate in the absorbing state.
So, the vector $\boldsymbol{\pi}'=(\pi_1,\ldots, \pi_p)$ describe the initial distribution of $\{X(t)\}_{t\in\Re_0^+}$ over the transient states only.

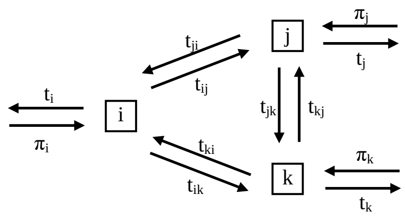
Definition
The distribution of the absorbing time τ

$$\tau = \inf \{t\geq 0 \mid X_t=p+1\}$$

is said to be a phase-type (PH) distribution with representation $(\boldsymbol{\pi}, \mathbf{T})$

$$\tau \sim \text{PH}(\boldsymbol{\pi}, \mathbf{T})$$

of order p.



The phase diagram of a phase type distribution with 3 phases, E={i,j,k}

Recalling that the matrix-exponential $e^{\boldsymbol{\Lambda}}$ is defined by the standard series expansion:

$$e^{\boldsymbol{\Lambda}} = \sum_{n=0}^{\infty} \frac{\boldsymbol{\Lambda}^n}{n!}$$

it is possible to show the following basic analytical properties of the phase-type distribution $\tau\sim\text{PH}(\boldsymbol{\pi}, \mathbf{T})$:

Theorem 1
The density function is: $f(x) = \boldsymbol{\pi}' \exp(\mathbf{T}x)\mathbf{t}$.

Theorem 2
The distribution function is: $F(x) = 1-\boldsymbol{\pi}' \exp(\mathbf{T}x)\mathbf{e}$.

Theorem 3

The n-th moment is: $E(\tau^n) = \int_0^\infty x^n dF(x) = (-1)^n n! \boldsymbol{\pi}' \mathbf{T}^{-n} \mathbf{e}$.

The moment generating function is: $E(e^{s\tau}) = \int_0^\infty e^{s\tau} dF(s) = \boldsymbol{\pi}'(-s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}$

(with $\mathbf{I}_{p\times p}$) .

Theorem 4

The Laplace-Stieltjes transform is: $\hat{F}[s] = \int_0^\infty e^{-s\tau} dF(s) = \boldsymbol{\pi}'(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}$ (with $\mathbf{I}_{p\times p}$) .

From theorem 2 derives that a phase-type distribution is light-tailed, since the tail of a phase-type distribution is exponentially decreasing.
Recalling that one of the advantages of using a phase-type distribution is that any distribution on positive axis can be well approximated by a phase-type distribution, from the last property it follows that for heavy-tailed distribution more attention is required.

4. Examples of phase-type distributions

By convenient choices of parameters, it is possible to obtain different distributions like exponential, Erlang, hiperexponential and Coxian.

<u>Example 1</u>
The random variable X~exp(λ) can be seen as a PH($\boldsymbol{\pi}$,**T**) with

$$\boldsymbol{\pi} = (1) \text{ and } \mathbf{T}=(\lambda)$$

So, the class of exponential distribution is the class of phase-type distributions with p=1.

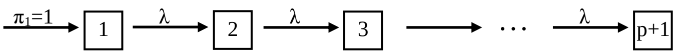
<u>Example 2</u>

The random variable $S_p = \sum_{k=1}^{p} X_k$, where X_k~exp(λ) are i.i.d., has an Erlang distribution.

The density function of S_p is obtained by a convolution of p exponential densities with the same parameter λ

$$f(x) = \lambda^p \frac{x^{p-1}}{(p-1)!} e^{-\lambda x} \ ,$$

and can be represented by the following phase diagram



Then, the distribution of S_p can be interpreted as a PH(**π**,**T**) with:

π'=(1, 0, …, 0), corresponding to E={1, …, p}

$$\mathbf{T} = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \lambda \end{pmatrix} .$$

If $S_p = \sum_{k=1}^{p} X_k$ has a generalized Erlang distribution, i.e. X_k~exp(λ_k), then S_p~PH(**π**,**T**) with

$$\mathbf{T} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{p-1} & \lambda_{p-1} \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_p \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \lambda_p \end{pmatrix} .$$

Representations of the Erlang random variable are by no means unique, because the X_k's can be summed in any order. So, alternative representations can be obtained permuting the states.

<u>Example 3</u>
Let X_k~exp(λ_k), with k=1, …, p, independent random variables. The hyperexponential distribution H_p is defined as a mixture of the p exponential distributions, with density:

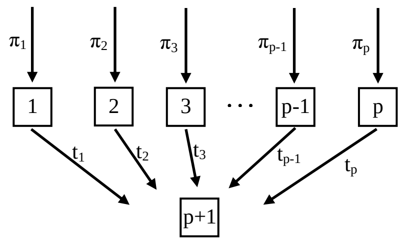
$$\sum_{k=1}^{p} \alpha_k \lambda_k e^{-\lambda_k x} \qquad \text{where } \alpha_k>0 \ (k=1, \ldots, p) \text{ and } \sum_{k=1}^{p} \alpha_k = 1.$$

Then, $H_p \sim PH(\boldsymbol{\pi}, \mathbf{T})$ with representation

$\boldsymbol{\pi}' = (\pi_1, \pi_2, \ldots, \pi_p)$

$$\mathbf{T} = \begin{pmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{p-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_p \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \vdots \\ \lambda_{p-1} \\ \lambda_p \end{pmatrix},$$

and the phase diagram is:



<u>Example 4</u>

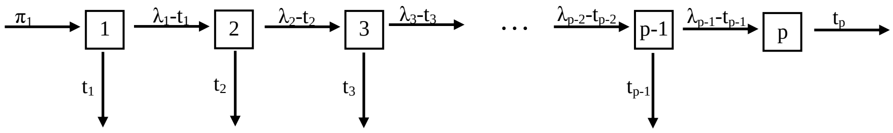
Let $X_k \sim \exp(\lambda_k)$, with $k=1, \ldots, p$, independent random variables. The Coxian distribution is defined as the sum S_N of a random number N ($N=1, \ldots, p$) of X_k.
The Erlang distribution is a special case of a Coxian distribution.
The class of Coxian distributions is interpreted as the class of phase-type distributions with representation:

$\boldsymbol{\pi}' = (1, 0, \ldots, 0)$, corresponding to $E=\{1, \ldots, p\}$

$$\mathbf{T} = \begin{pmatrix} -\lambda_1 & \lambda_1 - t_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 - t_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{p-1} & \lambda_{p-1} - t_{p-1} \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_p \end{pmatrix}$$

$$\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ \vdots \\ t_{p-1} \\ \lambda_p \end{pmatrix},$$

and phase diagram



5. Phase-type distributions in ruin theory

Phase-type distributions can be used in risk theory to model interarrival times as well as claim sizes.

Let Z_n ($n\geq1$) be a sequence of nonnegative independent random variables representing the interarrival times, or the time between the (n-1)th and n-th event (claim). Z_n have common distribution function $F(\cdot)$ and density $f(\cdot)$.

If $S_0=0$ and $S_n = \sum_{k=1}^{n} Z_k$ ($n\geq1$) it follows that S_n is the time of the n-th claim. The number of claims by time t is $N(t)=\max\{n \mid S_n\leq t\}$. The counting process $\{N(t)\}_{t\in\Re_0^+}$ is called a renewal process.

For applications in ruin theory, it is important to obtain the renewal density g(x) of $\{N(t)\}_{t\in\Re_0^+}$, which is the probability of a claim during the infinitesimal time interval [x, x+dx). Denoting by G(x) the renewal distribution function, it is

$$G(x) = \sum_{n=1}^{\infty} F^{*n}(x) \qquad\qquad g(x) = G'(x) = \sum_{n=1}^{\infty} f^{*n}(x)$$

The explicit calculation of the renewal density is usually not simple, but if $F(\cdot)$ is phase-type the problem has an analitically tractable solution.

It is possible to prove that if the interarrival times are phase-type with representation $(\boldsymbol{\pi},\mathbf{T})$, it follows that the renewal density is:

$$g(x) = \boldsymbol{\pi}' e^{(\mathbf{T}+\mathbf{t}\boldsymbol{\pi}')x}\mathbf{t} \ .$$

In fact, let $\{X^{(k)}(s)\}_{s\in\Re_0^+}$ be the Markov process governing the phase-type distribution of Z_k and define $\{J(s)\}_{s\in\Re_0^+}$ by joining the processes $\{X^{(k)}(s)\}_{s\in\Re_0^+}$:

$$\{J(s)\} = \{X^{(1)}(s)\}, \quad s \in [0, Z_1)$$

$$\{J(s)\} = \{X^{(2)}(s - Z_1)\}, \quad s \in [Z_1, Z_1 + Z_2)$$

$$\{J(s)\} = \{X^{(3)}(s - Z_1 - Z_2)\}, \quad s \in [Z_1 + Z_2, Z_1 + Z_2 + Z_3)$$

…..

$$\{J(s)\} = \{X^{(n)}(s - Z_1 - Z_2 - \cdots - Z_{n-1})\}, \quad s \in \left[\sum_{k=1}^{n-1} Z_k, \sum_{k=1}^{n} Z_k\right).$$

$\{J(s)\}_{s\in\Re_0^+}$ is a new Markov jump process on space state E with two types of transitions from i to j. One way is to jump following the process $\{X^{(k)}(s)\}$, at the rate $t_{ij}\in\mathbf{T}$, and the other way corresponds to a transition from $\{X^{(k)}(s)\}$ to the next $\{X^{(k+1)}(s)\}$ at rate $t_i\,\pi_j$. Hence, the intensity matrix of $\{J(s)\}_{s\in\Re_0^+}$ is $\mathbf{T}+\mathbf{t}\boldsymbol{\pi}'$ and the transition matrix of $\{J(s)\}_{s\in\Re_0^+}$ is $\exp(\mathbf{T}+\mathbf{t}\boldsymbol{\pi}')s$.

At time x the process $\{J(s)\}_{s\in\Re_0^+}$ develops through some process $\{X^{(k)}(s)\}_{s\in\Re_0^+}$. There is a renewal at time x if the phase-type process $\{X^{(k)}(s)\}_{s\in\Re_0^+}$ makes a transition to the absorbing state during [x, x+dx), so by the law of total probability the expression of the renewal density at x is

$$g(x) = \boldsymbol{\pi}' e^{(\mathbf{T}+\mathbf{t}\boldsymbol{\pi}')x}\mathbf{t}.$$

▪

Consider the classical Cramèr-Lundberg continuous time risk model that could be regarded as a particular case of a renewal (Sparre Andersen) model.

Let N(t) be the number of claims from an insurance portfolio. It is assumed that N(t) $(t\geq 0)$ follows a Poisson process with mean λ. The individual claim sizes, U_1, U_2, ... independent of N(t), are positive, independent and identically distributed random variables with $P(x)=\Pr\{X\leq x\}$ distribution function and $p(x)=dP(x)$ individual claim amount probability density function.

The insurer's surplus process at time t $(t\geq 0)$ is

$$W(t) = u + ct - \sum_{j=1}^{N(t)} U_j$$

where u=W(0)≥0 is the insurer's initial surplus, c the premium rate per unit time.
The time of ruin T is the first time that the surplus becomes negative defined by

$$T = \inf \left\{ t \,\middle|\, W(t) < 0 \right\} ,$$

with T=+∞ if W(t)≥0 for all t≥0 (i.e., if ruin does not occur).
The probability of ultimate ruin as function of the initial surplus u is

$$\psi(u) = \Pr\left\{ T < +\infty \,\middle|\, W(0) = u \right\} .$$

In literature it is well known that obtaining an explicit formula for ψ(u) is not simple. In fact, only for particular distributions of individual claim amount it is possible to find an exact solution.
The class of phase-type distributions is the one within computationally tractable exact forms of the ruin probability ψ(u) can be obtained.
In the hypotheses of phase-type distribution for individual claim size with representation (**π**,**T**)

$$P(x) = 1 - \boldsymbol{\pi}' e^{\mathbf{T}x}\mathbf{e} \qquad\qquad p(x) = \boldsymbol{\pi}' e^{\mathbf{T}x}\mathbf{t}$$

it is possible to show that

$$\psi(u) = \boldsymbol{\pi}'_{-}\, e^{(\mathbf{T}+\mathbf{t}\boldsymbol{\pi}'_{-})u}\mathbf{e} \; .$$

The i-th component of the vector $\boldsymbol{\pi}_{-}$ is the probability that a Markov jump process underlying the phase-type claims downcrosses level u in state i when the surplus process jumps to a level below u for the first time. Since there is a positive probability that $\{W(t)\}_{t\in\Re_0^+}$ never goes to a level below u, the distribution $\boldsymbol{\pi}_{-}$ is defective.

In this case, when the claims are phase-type, also the process underlying the descending ladder heights is a terminating phase-type renewal process with interarrival distribution PH($\boldsymbol{\pi}_{-}$, **T**).

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